Fundamentals of Digital Commun. Ch. 4: Random Variables and Random Processes

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Outline

- 4-1 Probability (brief review)
- 4-2 Random Variables
- 4-3 Random Processes
- 4-4 Noise – Example for a Random Process

4-1 Probability

- Fundamental concept:
  - random experiment with an **uncertain outcome**
- Definitions:
  - $\omega \ldots$ outcomes: $\omega \in \Omega$
  - $\Omega \ldots$ sample space: set of all outcomes
  - **Events:** subsets of sample space Ω for which *propability* can be defined
  - **Probability:** functional mapping of events onto reals: $0 \leq P(E) \leq 1$ for all events $E \in \mathcal{B}$
  - $\mathcal{B} \ldots$ collection of (all possible) events
  - $(\Omega, \mathcal{B}, P) \ldots$ probability space

Probability (cont'd)

- Probability is:
  - relative frequency of an event ocurring in n trials
  - A, B ... events

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

  - n_A ... number of occurancies of A
  - n ... number of trials
- definitions
  - sure event E: $P(E) = 1$
  - null event $A = 0$: $P(A) = 0$

Probability (cont'd)

Events can be represented as sets (Venn diagram)

- product of events "A and B" (intersection)

$$C = A \cap B = AB$$

- mutually exclusive events $A \cap B = 0$, i.e. $P(AB) = 0$
- sum of events "A or B" (union)

$$C = A \cup B = A + B$$
$$P(A + B) = P(A) + P(B) - P(AB)$$

Probability (cont'd)

- conditional probability

$$P(A|B) = \lim_{n\to\infty} \frac{n_{AB}}{n_B} = \frac{P(AB)}{P(B)}$$

  - *probability that an event A occurs, given that an event B has also occured*

- product theorem

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

- leads to the **Bayes-Theorem**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Probability (cont'd)

- total probability:
  - *event B occurs jointly with one of the mutually exclusive events $A_1, A_2, ..., A_K$,* where

$$\sum_{k=1}^{K} P(A_k) = 1$$

  - *Find the probability $P(B)$,* given $P(B|A_k)$ and $P(A_k)$:

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... \\ &\quad + P(B|A_K)P(A_K) \\ &= \sum_{k=1}^{K} P(B|A_k)P(A_k) = \sum_{k=1}^{K} P(BA_k) \end{aligned}$$

Probability (cont'd)

- Def.: a-priori probabilities: $P(A_k)$
- Def.: a-posteriori probability: $P(A_k|B)$
  - *Assume: event B occurs jointly with one of the mutually exclusive events $A_1, A_2, ..., A_K$:*
  - *Find the probability for A_k, when B has already occured. (B could be an observation of A_k)*

  can be solved using the Bayes-theorem and the total probability

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{K} P(B|A_i)P(A_i)}$$

  given $P(A_k)$ and $P(B|A_k)$ (likelihoods)

4-2 Random Variables

- sample space Ω (with outcomes ω_i, ν_j (and events)) is **mapped** onto the set of reals $\mathbb{R}$
  the random variable is a function of the outcomes
  - notation: $X = X(\omega_i), Y = Y(\nu_j)$, short $\boxed{X, Y}$
    upper-case symbols are used to distinguish from "defined" variables
- discrete random variables (RV)
  e.g. dice $X \in \{1, 2, ..., 6\}$ (number of eyes);
  $P(X = x_m) = \frac{1}{6}$
- continuous RV
  e.g. voltage of a battery $X = U$

Characterization of RVs

- **cumulative distribution function** – CDF
  (dt: Verteilungsfunktion)

$$F_X(a) = P(X \leq a) = \lim_{n \to \infty} \frac{n_{(X \leq a)}}{n}$$

  dimensionless

- **probability density function** – PDF
  (dt: Wahrscheinlichkeitsdichtefunktion)

$$f_X(x) = \frac{dF_X(x)}{dx} = \lim_{n \to \infty, \Delta x \to 0} \left[\frac{1}{\Delta x} \left(\frac{n_{\Delta x}}{n} \right) \right]$$

  unit is $1/[x]$

Characterization of RVs (cont'd)

- properties of the CDF
  - $0 \leq F_X(x) \leq 1$, nondecreasing
  - $F_X(-\infty) = 0$
  - $F_X(\infty) = 1$
  - following the def.: discrete points at $X = x$ are included
- probability that X lies in some interval:

$$\begin{aligned} P(x_1 < X \leq x_2) &= P(X \leq x_2) - P(X \leq x_1) \\ &= F_X(x_2) - F_X(x_1) \end{aligned}$$

Characterization of RVs (cont'd)

- properties of the PDF
  - PDF is **nonnegative** $f_X(x) \geq 0$
  - area under the PDF $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- computation of the CDF from the PDF

$$F_X(x) = \int_{-\infty}^{x+\epsilon} f_X(\lambda)d\lambda = P(X \leq x)$$

- probability that X lies in some interval:

$$P(x_1 < X \leq x_2) = \int_{x_1+\epsilon}^{x_2+\epsilon} f_X(\lambda)d\lambda$$

Ensemble Averages and Moments

- mean value, expected value, first-order moment, ensemble average

$$\mathrm{E}\{X\} = m_X = \int_{-\infty}^{\infty} x f_X(x) dx$$

(center of gravity of the PDF)

- k-th order moment

$$\mathrm{E}\{X^k\} = \int_{-\infty}^{\infty} x^k f_X(x) dx$$

- centralized k-th-order moment (moment about mean)

$$\mathrm{E}\{(X - m_X)^k\} = \int_{-\infty}^{\infty} (x - m_X)^k f_X(x) dx$$

Ensemble Averages and Moments (cont'd)

- variance (centralized second moment)

$$\sigma_X^2 = \mathrm{var}\{X\} = \mathrm{E}\{(X - m_X)^2\} = \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x) dx$$

- standard deviation

$$\sigma_X = \sqrt{\mathrm{var}\{X\}}$$

- generalized: mean of a function $g(X)$ of an RV

$$\mathrm{E}\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Ensemble Averages and Moments (cont'd)

- definition through events (for lim $n \to \infty$)

$$\mathrm{E}\{X\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X(A_i)$$

$$\mathrm{E}\{g(X)\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g[X(A_i)]$$

- can be used to **estimate** moments
- will be an approximation for a limited sample $n < \infty$

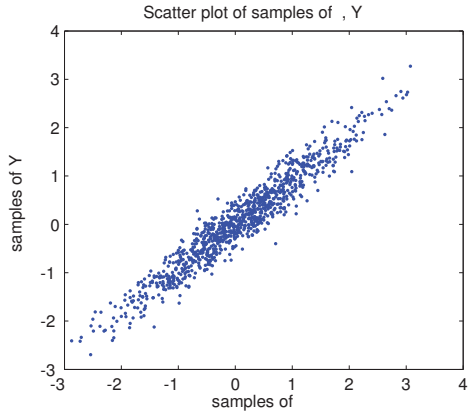
Extension to two Random Variables

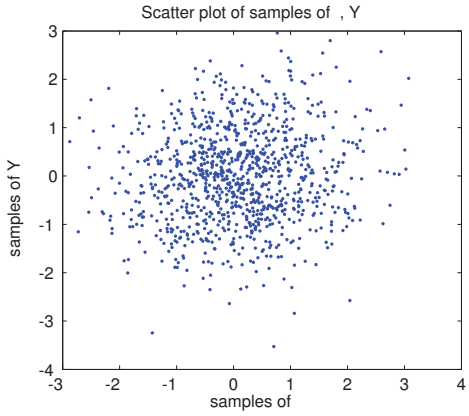
- bivariate statistics: joint characterization of two RVs
  - for > 2 variables: multivariate statistics
- CDF

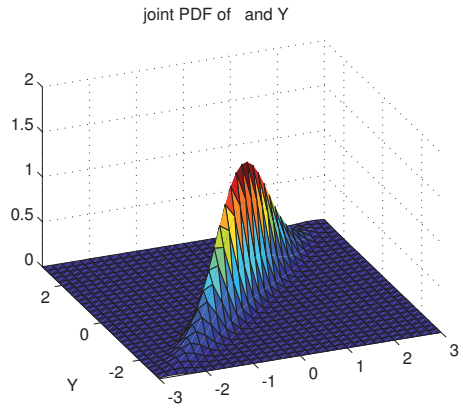
$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

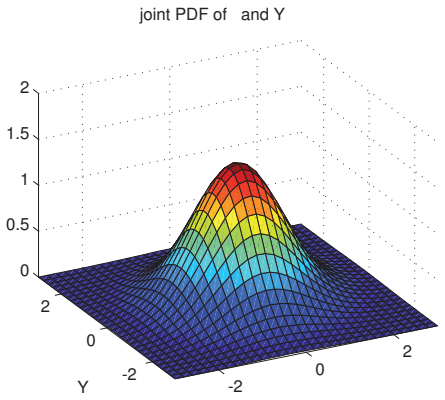
- PDF

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$









Extension to two Random Variables (cont'd)

- **correlation**

$$R_{XY} = \mathrm{E}\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dx\,dy$$

- uncorrelated RVs

$$R_{XY} = \mathrm{E}\{XY\} = \mathrm{E}\{X\}\mathrm{E}\{Y\} = m_X m_Y$$

- orthogonal RVs

$$R_{XY} = \mathrm{E}\{XY\} = 0$$

- **covariance**

$$K_{XY} = \mathrm{E}\{(X - m_X)(Y - m_Y)\} = R_{XY} - m_X m_Y$$

4-3 Random processes

- $X(A_i, t)$ ... function of **two** variables:
  - A_i ... event, realization
  - t ... time (usually)
- for an event A_i: sample function $x_i(t)$
- **ensemble** of sample functions: $\{x_i(t)\}$, $\forall i$
  - **random process**: $X(t)$
    (also in lower case)
- illustration of an ensemble of sample functions:
  (blackboard)

Characterization of random processes

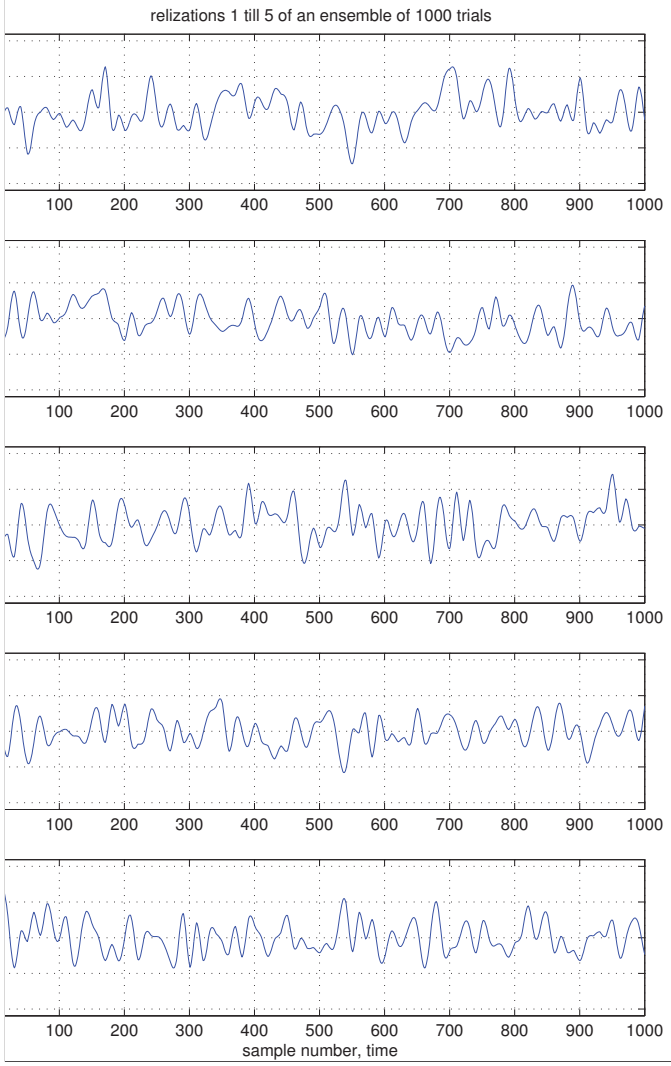
- ensemble averages
  - mean (at time t)

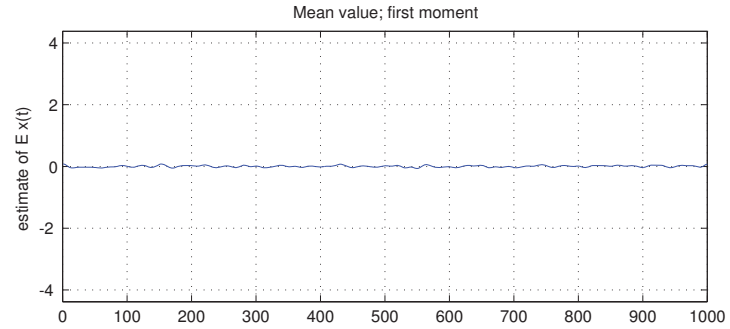
$$m_X(t) = \mathrm{E}\{X(t)\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx$$

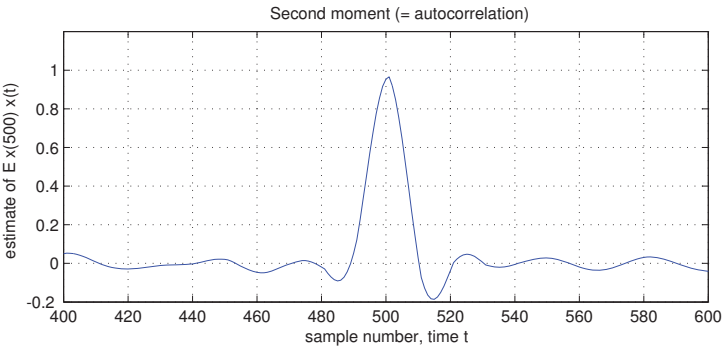
    can be **time-variant** (e.g. sine-wave plus noise)
  - autocorrelation(function – ACF)

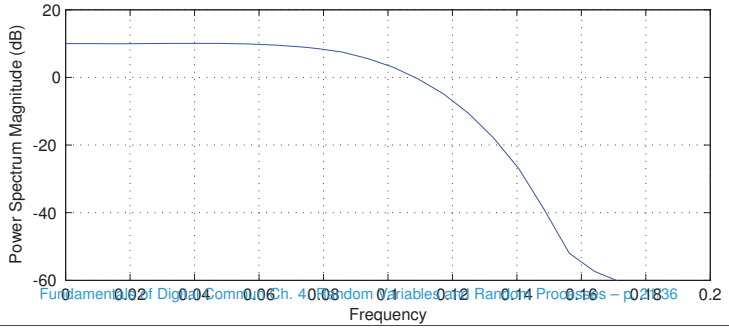
$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1, x_2)\, dx_1 dx_2$$

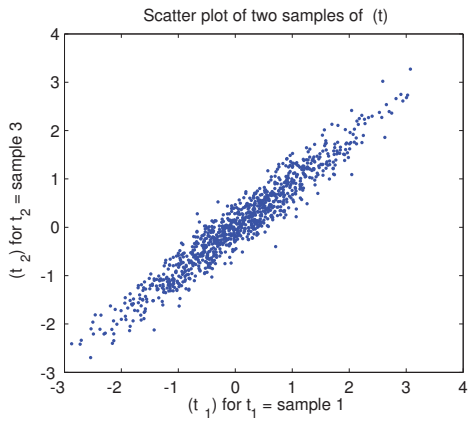
$f_{X(t_1)X(t_2)}(x_1, x_2)$ ... joint PDF for $X(t_1)$ and $X(t_2)$

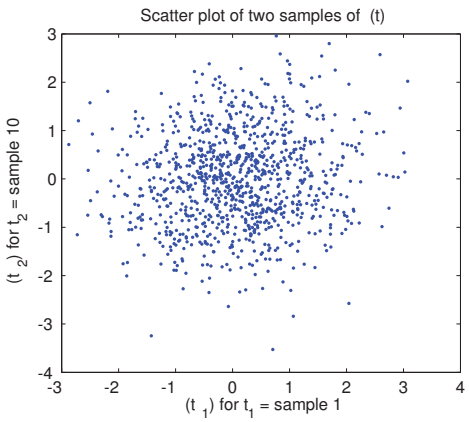

relizations 1 till 5 of an ensemble of 1000 trials
sample number, time

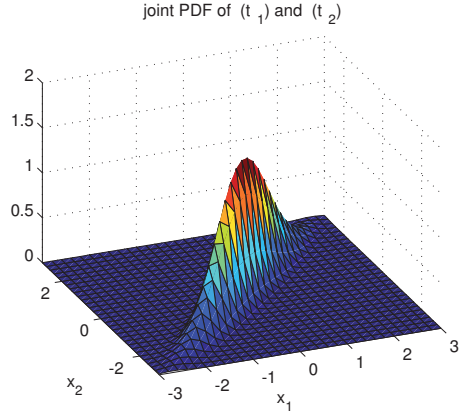

Mean value; first moment
estimate of E x(t)

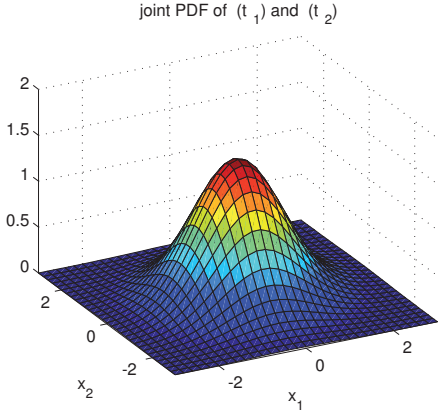

Second moment (= autocorrelation)
estimate of E x(500) x(t)
sample number, time t


Power Spectrum Magnitude (dB)
Frequency


Scatter plot of two samples of (t)
(t 2) for t2 = sample 3
(t 1) for t1 = sample 1


Scatter plot of two samples of (t)
(t 2) for t2 = sample 10
(t 1) for t1 = sample 1


joint PDF of (t 1) and (t 2)
x2
x1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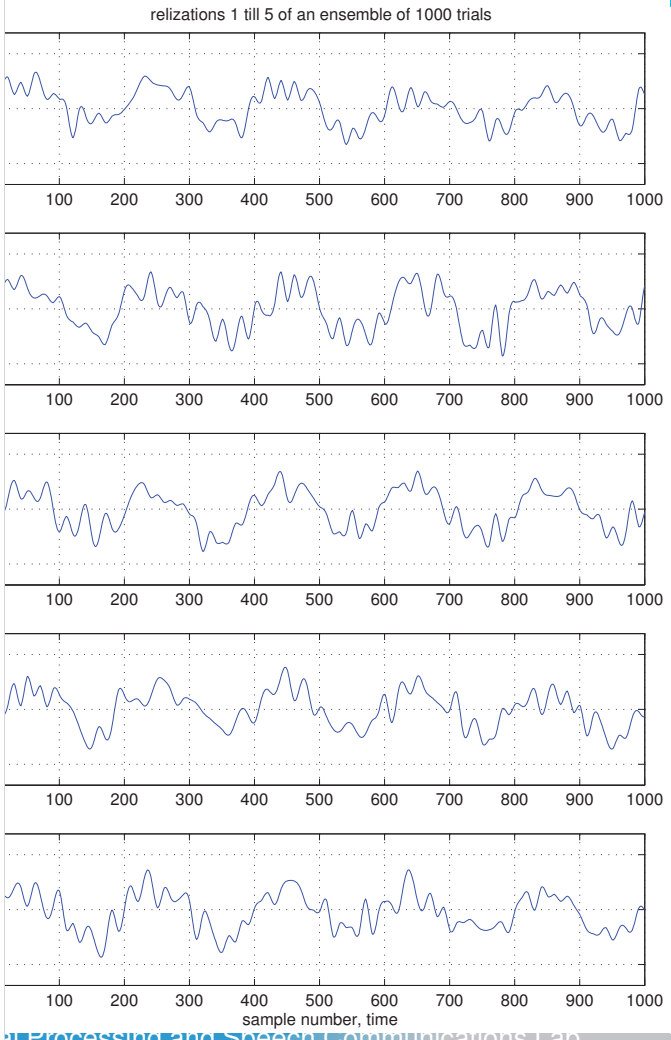
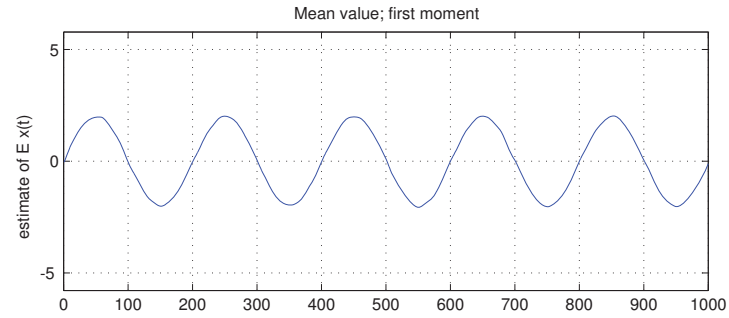
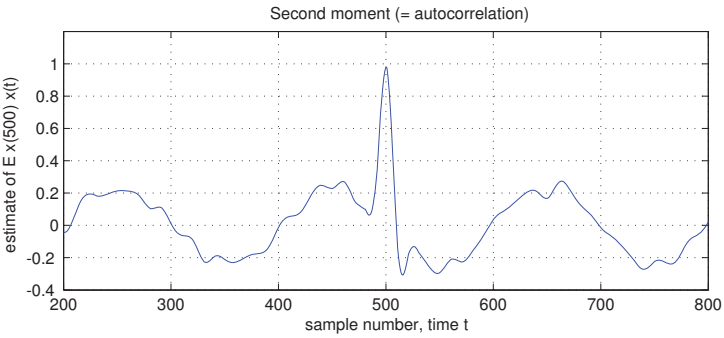
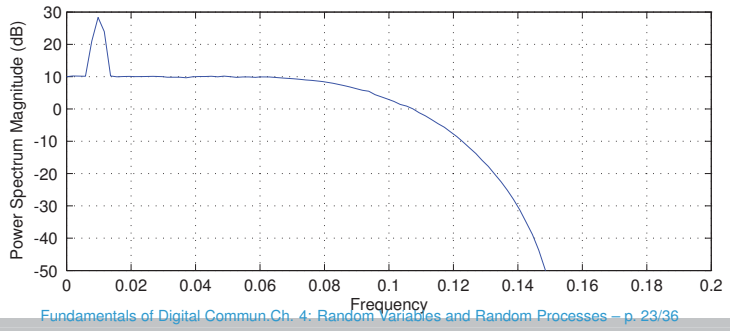








Characterization of random processes (cont'd)

- stationarity (time-**in**variance)
  - first-order stationarity (stationary w.r.t. mean)

$$\mathrm{E}\{X(t)\} = m_X = \text{const.}$$

  - second-order stationarity (stationary w.r.t. ACF)

$$R_X(t_1, t_2) = R_X(\tau) = \mathrm{E}\{X(t)X(t+\tau)\}$$

    where $\tau = t_1 - t_2$
    i.e.: no variability w.r.t. time shift

- exact definition:
  Stationarity (of n-th order): PDF (of n-th order) is invariant w.r.t. time shifts

Characterization of random processes (cont'd)

Important definitions:

- Strict-Sense-Stationary (SSS) (Stationär im strengen Sinn)
  - PDFs of **any order** are stationary
- Wide-Sense-Stationary (WSS) (Stationär im weiteren Sinn)
  - Mean and autocorrelation function (ACF) are stationary
    (i.e. first and second-order moments)
  - **for Gaussian processes**: WSS implies SSS (hence the importance of the first two moments)

Characterization (cont'd); ACF of WSS Random Processes

- Properties
  - $R_X(\tau) = R_X(-\tau)$ ... even symmetry
  - $R_X(0) \geq |R_X(\tau)|$
  - $R_X(0) = \mathrm{E}\{X^2(t)\}$ ... second moment; $\propto$ **power**
- Power Spectral Density – PSD (Leistungsdichtespektrum)
  - $X(t)$ ... **power signal**, if stationary
  - PSD is Fourier transform of the ACF

$$\text{Def.}: \quad R_X(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_X(j\omega)$$

  - PSD is power content in frequency domain

Power Spectral Density (PSD) (cont'd)

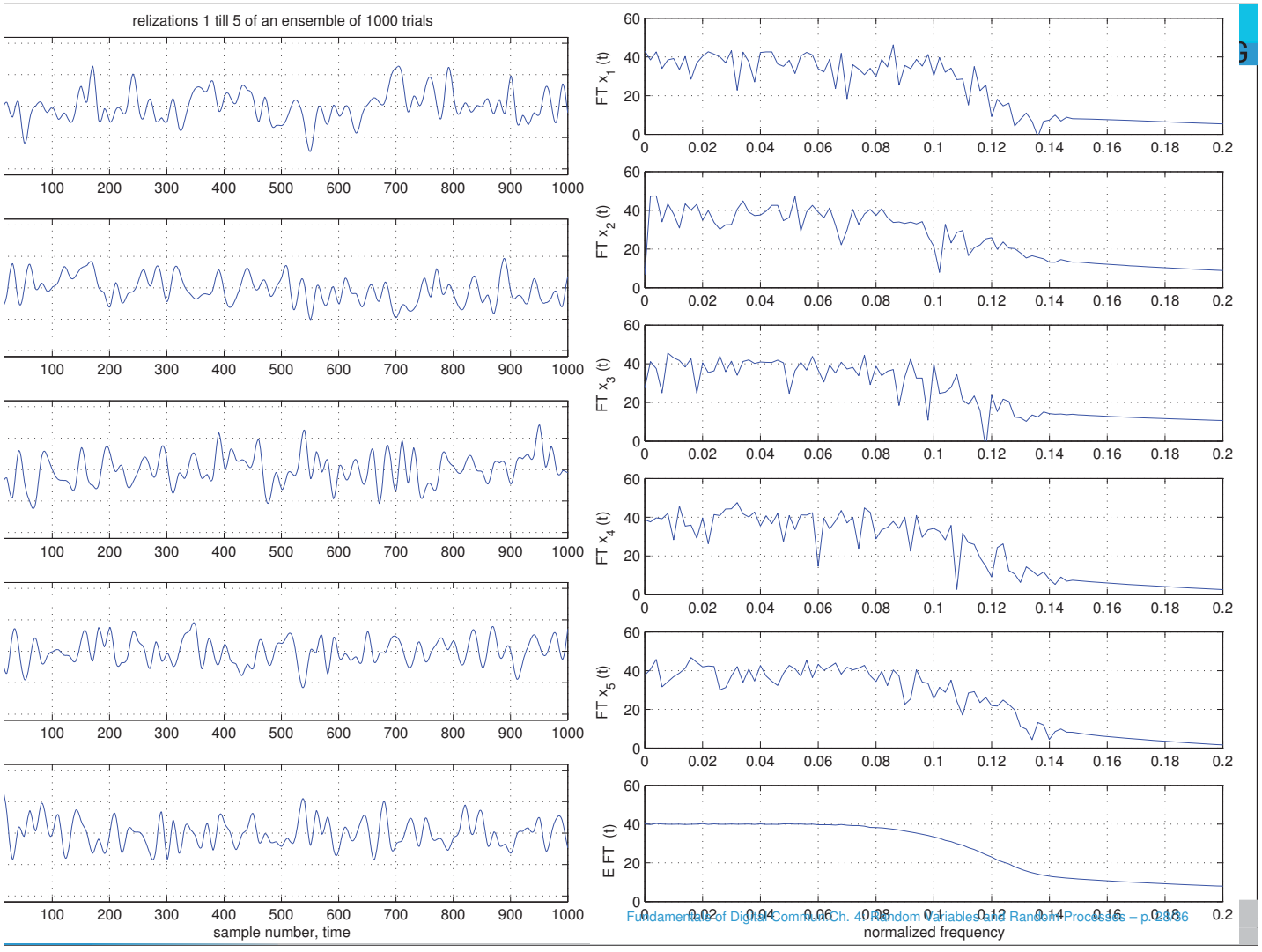
- definition using the Fourier transform of $X(t)$:
  - needs time limitation of the sample functions:

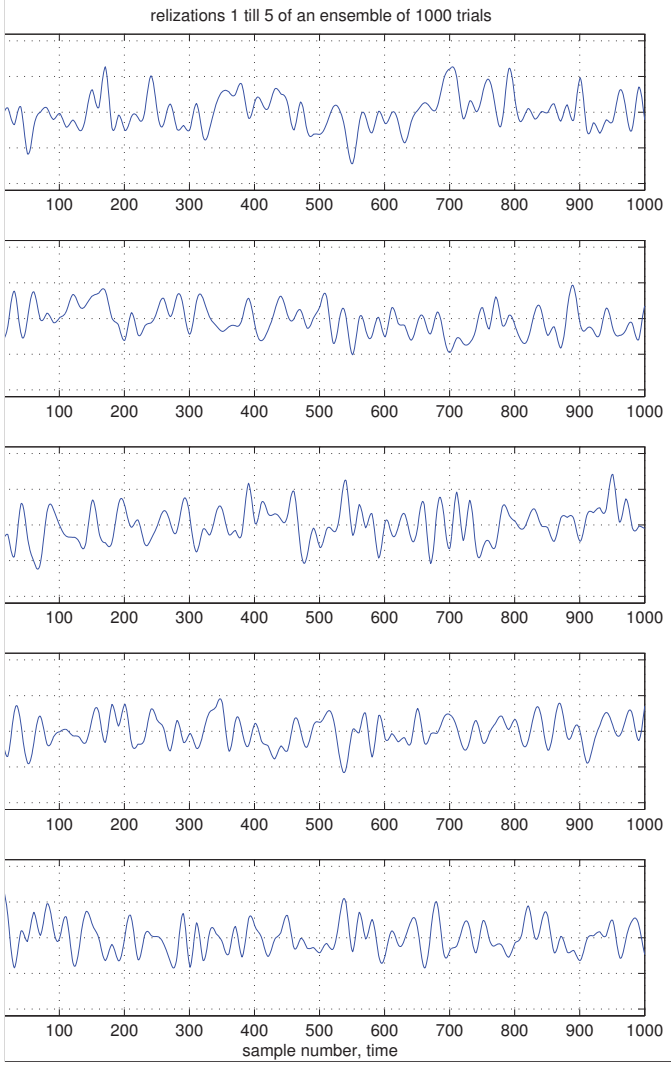
$$X_T(A_i, j\omega) = \int_{-T/2}^{T/2} x(A_i, t) e^{-j\omega t}\, dt$$

- Wiener-Kinchin theorem

$$S_X(j\omega) = \lim_{T\to\infty} \frac{1}{T} \mathrm{E}_{A_i}\{|X_T(A_i, j\omega\}|^2\}$$
$$= \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau}\, d\tau = \mathcal{F}[R_X(\tau)]$$

Proof at blackboard!

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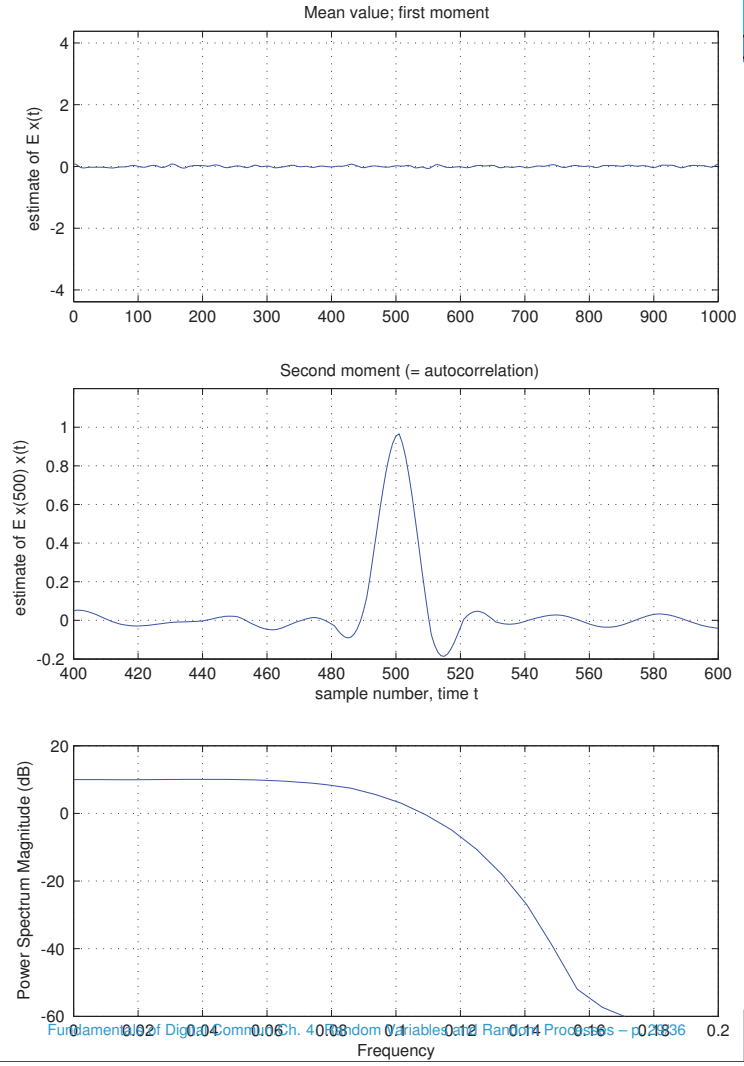
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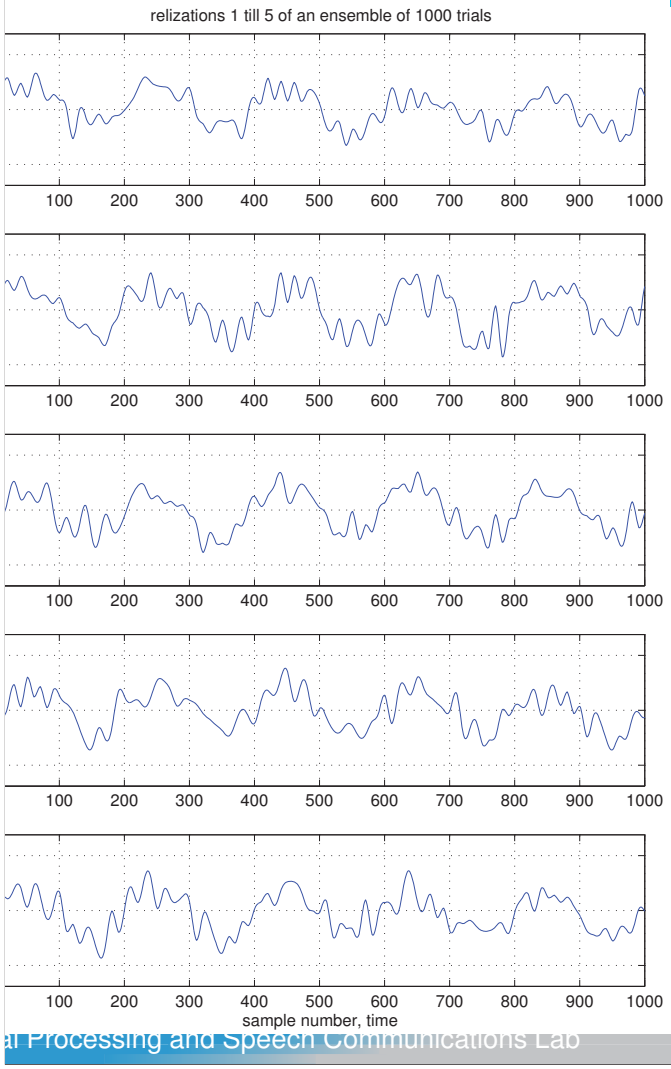
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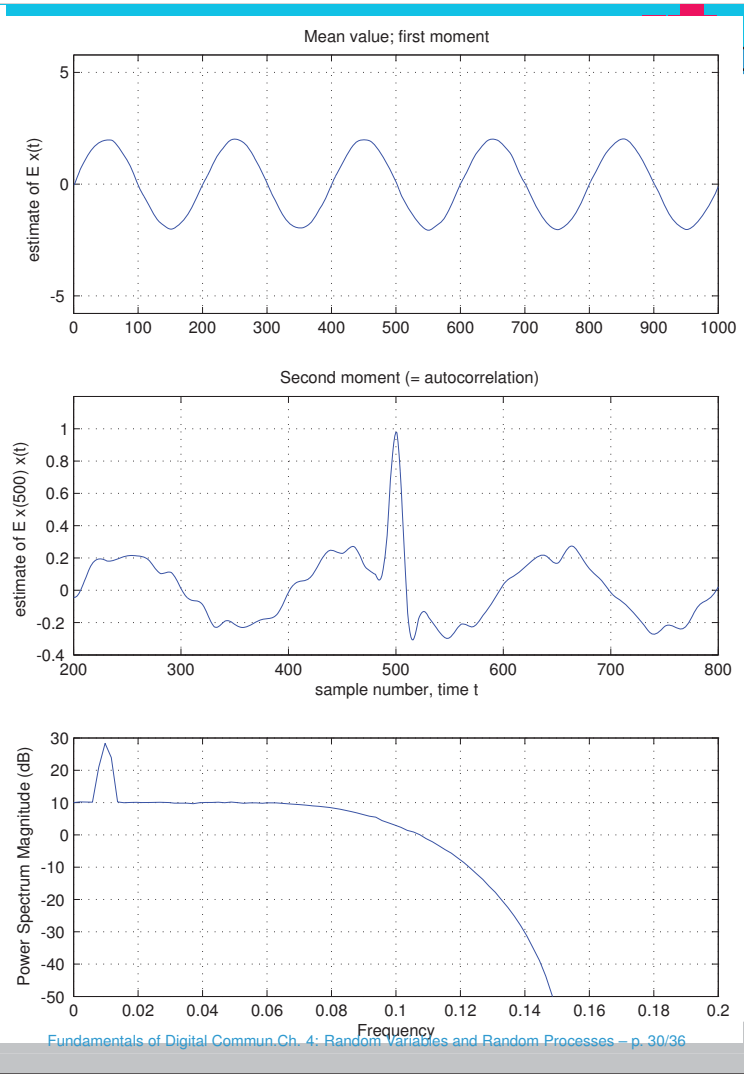
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Characterization (cont'd); Ergodic Random Proc.

- Definition: (**all**) time and ensemble averages are identical
  - necessary condition: **stationarity** (otherwise, ensemble average would be time-variant)
  - example: ensemble average and DC-value it must hold:

$$x_{DC} \equiv m_X \text{ , where}$$

$$x_{DC} = \langle x_i(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)\, dt$$

$$m_X = \mathrm{E}\{X(t)\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i(t)$$

Characterization (cont'd); Ergodic Random Proc.

- equivalence must hold for all averages → hard to proof
- Example: ACF

$$R_X(\tau) = \mathrm{E}\{X^*(t)X(t+\tau)\} \equiv \langle x_i^*(t)x_i(t+\tau)\rangle$$

- **consequence** of ergodicity: ensemble averages (moments) can be estimated from a single sample function!

4-4 Noise

- Example for a random process
- Physics: freely moving electrons e^-
- PSD, ACF

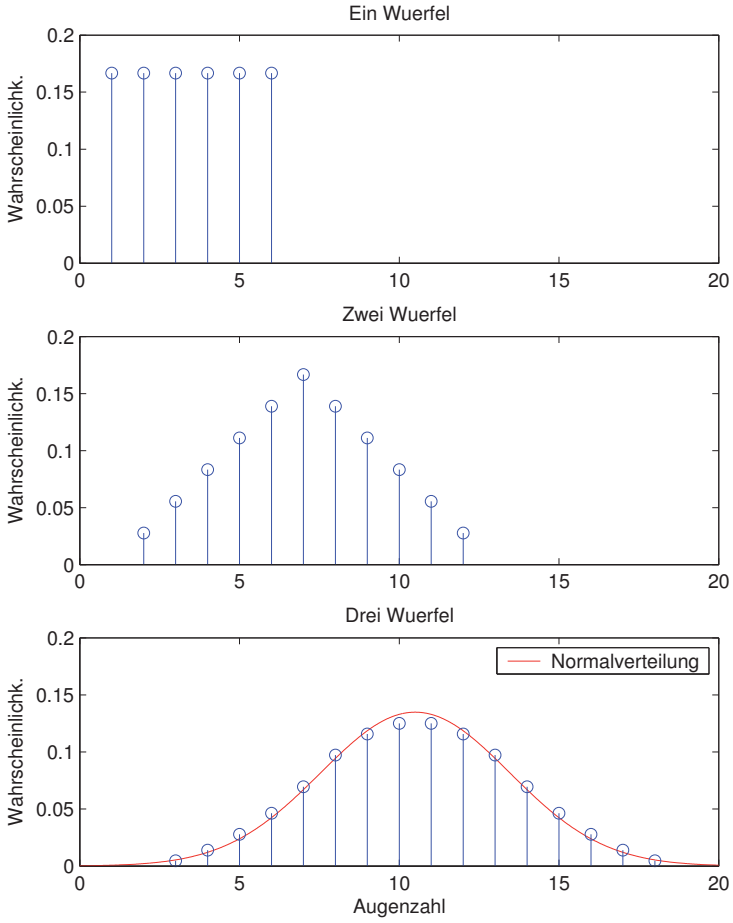
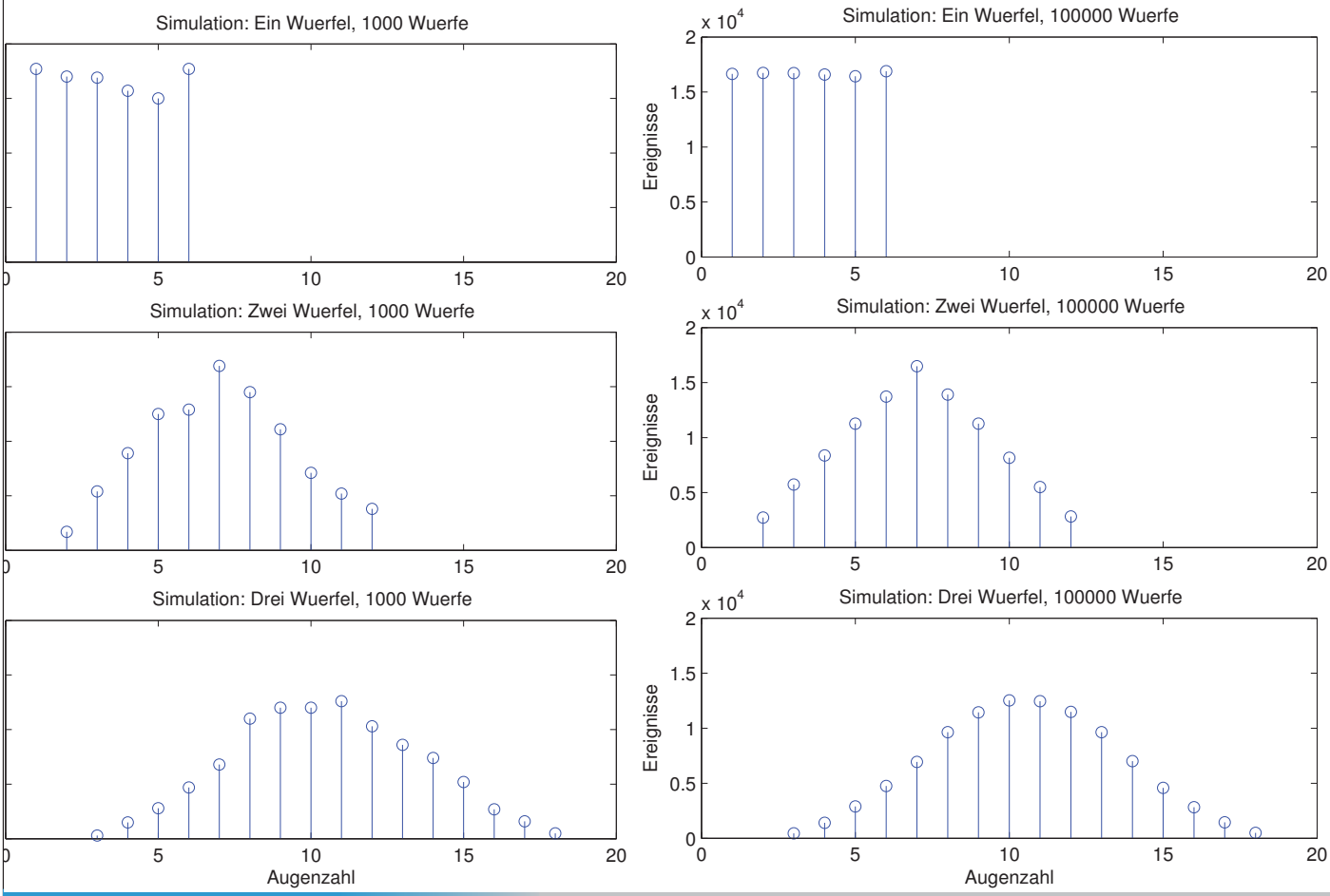
$$S_X(j\omega) = S_X(f) = \frac{N_0}{2} \overset{\mathcal{F}}{\longleftrightarrow} R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

  $N_0/2$ ... double-sided noise PSD
- thermal noise $N_0 = kT$
- PDF: Gaussian PDF = **normal** distribution
  - Power: $P_N = N_0 B = \sigma^2$; B ... equivalent bandwidth

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

Why is noise Gaussian distributed?

- **Central Limit Theorem** (Zentraler Grenzwertsatz): a sum of (many) random processes has Gaussian PDF
- illustration:
  - PDF of one dice: uniform (discrete)
  - two dice: triangular PDF
  - many dice: PDF approaches Gaussian PDF
- generally (without proof): PDF of the sum of two (independent) random variables corresponds to the **convolution** of their PDFs



(discrete PDFs)