Maximally Decimated Filterbanks

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Contents

1 Introduction And Some Terms

The need for filter banks is quite obvious in modern signal processing systems. Generally known is the use in audio and speech coding, but it is also essential to accelerate adaptive systems or convolution algorithms. So a lot of research has been done in the past and is still going on. This article should present a brief introduction/tutorial to the theory of the analysis and synthesis combination called "Maximally Decimated Filter Bank". This term implies that the samplerates of each subband shall be reduced to the minimum. The bandwidth after the downsampling of each band is just 2 times the Nyquist Frequency. So, the overall data rate remains the same. Another familiar term is "Quadrature Mirror Filter". Originally this is a special early form of a 2-Channel Maximally Decimated Filter Bank. The main concept is that aliasing is permitted between the analysis and synthesis filter but is canceled out at the output. Today this term is used more generally for M-Band Maximally Decimated Filter Banks which incorporate this attribute.
At first we will start with the theory of construction for 2-Channel filterbanks. Later we will expand this theory to an arbitrary number of channels.

2 2-Channel QMF-Banks

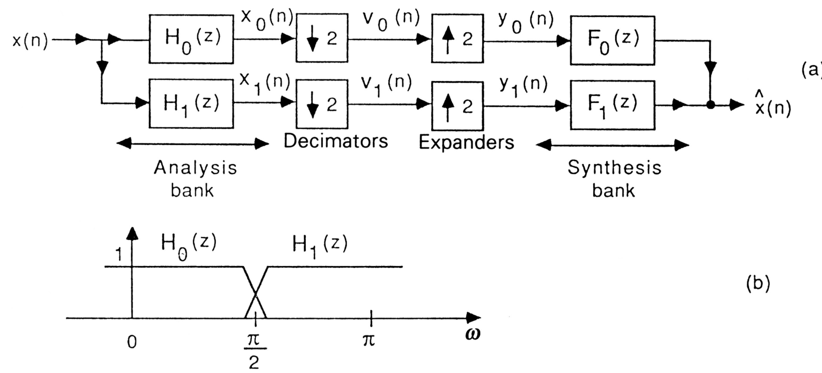

Figure 5.1-1 (a) The quadrature mirror filter bank and (b) typical magnitude responses.

Figure 1: QMF Structure [Vaid93]

The structure of the filter banks in this chapter is given as above. The task is to design the analysis filters H_0, H_1 and the synthesis filters F_0, F_1.

2.1 Errors Created In QMF-Banks

In a filter bank the following degradations are possible. According to the design principles one or more of them should be omitted by a correct design of the filters.

Aliasing
This is perhaps the most annoying artefact in here, because Aliasing is a nonlinear, non reversible process. It is perceptually very nasty and may be the cause of some strange, unexpected errors. The problematic is quite similar as in all AD/DA-tasks. Conventionally it must be secured, that there is no signal-energy beyond π. Ideally with no degradations of the remaining signal.

Amplitude Distortions
This is most obvious. If for example the two analysis filters have a large gap in between them, a lot of frequencies will have very few gain in comparison to others. In practice it will be hard to restore the lost signal content because the SNR is very low at these frequencies.

Phase Distortions
Most IIR and also FIR Filters produce severe nonlinear phase distortions which will remain after reconstruction.

Coding and Quantization Artifacts
Although these Degradations have nothing to do with the filter design task it is quite necessary to mention it here. Coding and Quantization is often applied in transmission systems. In the precedent graph this operation will occur in between the decimators and expanders. These artifacts can be

influenced by the filter design. Unfortunately, there is no way to calculate these effects analytically. It is just possible to make some statistical estimations and/or to reduce them with numerical schemes.

To illustrate the practical considerations have a look at figure 2.

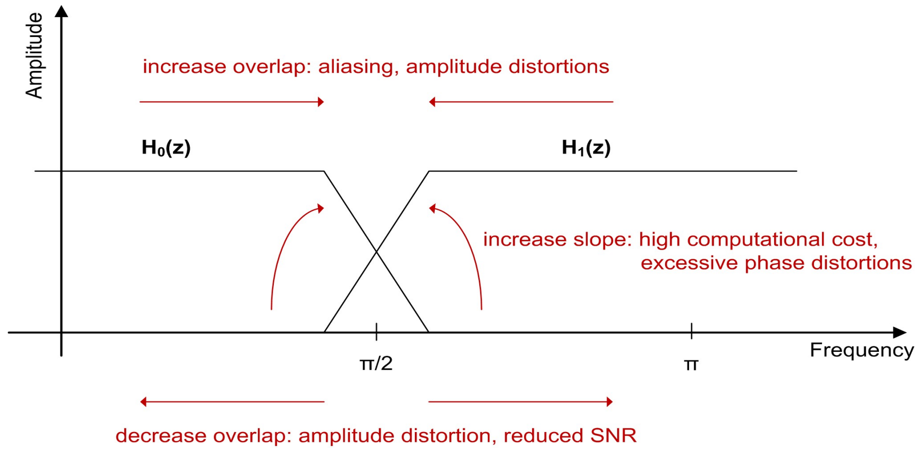

Figure 2: Filter Design Tradeoffs

The ideal solution would be a rectangular highpass and lowpass. In reality this is not possible because this attempt will result in an infinite impulse response. So a designer is forced to make some tradeoffs. Given two trivial filters he may at first vary their cutoff frequencies. If he increases the overlap, the sum of the two filters may produce gains of more than 1 at some frequencies. More dramatic is the increased aliasing, because it is not possible to increase the subband sampling frequencies. If he decreases the overlap on the other hand, the aliasing will also decrease, but we have a large frequency gap at $\pi/2$. Eliminating these gaps by boosting these frequencies will fail because it would just amplify noise there and reduce the SNR. Increasing the slopes will reduce the problems, but only with noticeable computational costs. Additionally, most efficient and steep IIR filters do add severe phase distortions.

So, to solve this task analytically,it is necessary to describe the system first by transfer functions:

$$\begin{aligned}
X_k(z) &= H_k(z)X(z) && ,k=0,1 \\
V_k(z) &= \frac{1}{2}[X_k(z^{\frac{1}{2}}) + X_k(-z^{\frac{1}{2}})] && \mathit{Aliasing}!!! \\
Y_k(z) &= V_k(z^2) = \frac{1}{2}[H_k(z)X(z) + H_k(-z)X_k(-z)] && \mathit{Aliasing}!!! \\
\hat{X}(z) &= F_0(z)Y_0(z) + F_1(z)Y_1(z) && \\
\hat{X}(z) &= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) && \\
&\quad + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z) && \mathit{Aliasing}!!! \\
A(z) &= \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)] \rightarrow 0 && \mathit{Aliasing\ TF} \\
T(z) &= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)] && \mathit{Distortion\ TF}
\end{aligned}$$

This representation is very advantageous, because it describes the amplitude and phase characteristics of the system in one term (*Distortion TF* $T(z)$) and the aliasing components in a separate term (*Aliasing TF* $A(z)$). The second term enters the system at the decimator ($V_k(z)$) and furthermore never disappears.
Every shortcoming of the system can be extinguished step by step.

The First Step: Remove Aliasing.

So have a look at the Aliasing Transfer Function $A(z)$. The transfer function incorporates both the analysis and synthesis stage. The first impulse would be to reduce the aliasing at the decimation point to a minimum by using "ideal" filters. As mentioned before, this would be quite expensive and would lead to very rigid restrictions for the filter construction before even having had a look at the complete great project in which the filter bank is just a small part.
The direct, plain and successful approach states: "Don't mind if there's aliasing, if it doesn't leave the output! Just set A(z) equal to zero." The filters H_0 and F_0, H_1 and F_1 have to be designed in a way, so that the aliasing signals of each subchannel cancel each other out in the last stage. In this case, the aliasing disappears completely. Figure 3 illustrates this idea.

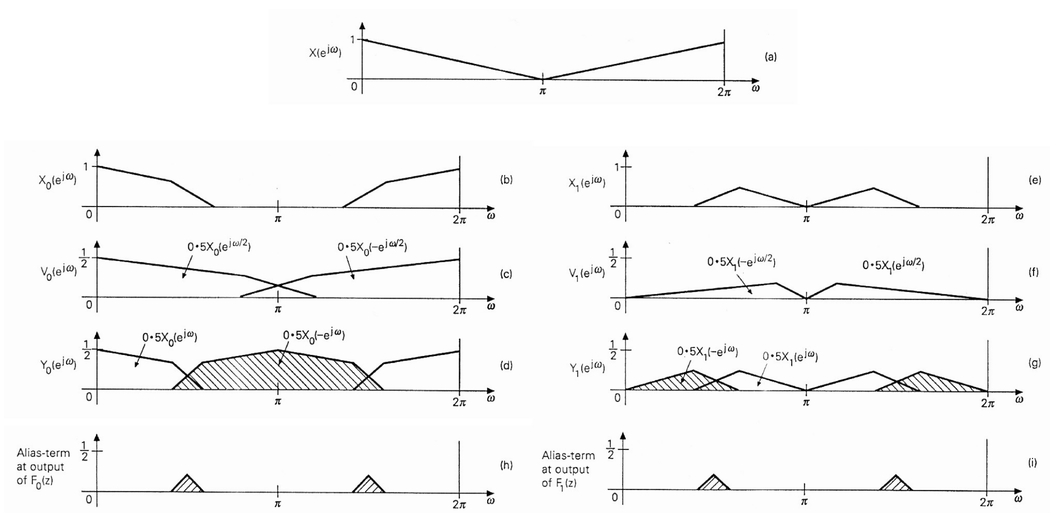

Figure 5.1-3 Various internal signals, and alias cancelation mechanism in the QMF bank. (© Adopted from 1990 IEEE.)

Figure 3: Aliasing Cancellation [Vaid93]

So, the aliasing is gone. The transfer function of the whole system is reduced to the Distortion Transfer Function:

$$\hat{X}(e^{j\omega}) = |T(e^{j\omega})|e^{j\phi\omega}X(e^{j\omega})$$

The Second Step: Remove The Objectionable Distortions

The question is now: Is it sufficient to remove the phase distortions, the amplitude distortions or both? Depending on the aim there are three different target formulations:

- **Free from Phase Distortion:**

$$\phi(\omega) = a + b\omega, \quad a, b = const$$

  In words: the TF just consists of linear phase components.

- **Free from Amplitude Distortion:**

$$|T(e^{j\omega})| = d, \quad d \neq 0 \; for \; all \; \omega$$

  No matter what the phase characteristics are the amplitude in the frequency domain is constant. The whole system is a allpass filter.

- **Perfect Reconstruction (PR):**

$$T(z) = cz^{-m_0}, \quad c = const$$

  The impulse response of the output is just a time shifted version of the input.

2.2 "Classic" QMF

In this chapter some solutions for the demands formulated in section 2.1 are shown. Look at the whole transfer function and begin with step one: the aliasing cancellation.

$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) \\ &\quad + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z)\end{aligned}$$

A very simple and straightforward solution is the following (by the way, it is not the only possible one):

$$\begin{aligned}H_1(z) &= H_0(-z) \\ F_0(z) &= H_0(z) \\ F_1(z) &= -H_1(z) = -H_0(-z)\end{aligned}$$

So $H_1(z)$ is a (at $\pi/2$) mirrored Version of $H_0(z)$. The only challange is to design one filter, a lowpass for example and all other filters are fixed. This reduces the complexity of all further steps to a great extend. The resulting Aliasing and Distortion Transfer Functions are:

$$\begin{aligned}T(z) &= \frac{1}{2}[H_0^2(z) - H_0^2(-z)] \\ A(z) &= 0\end{aligned}$$

The Aliasing is cancelled out as desired, the Distortion Transfer Function is quite a short term. Regarding its appearance, it's quite clear why this filter is called "Quadrature Mirror Filter".

2.2.1 Eliminating Phase Distortion

To be free of phase distortion the whole system must behave as a linear phase filter. This is the case, if H_0 is linear phase:

$H_0(z) =$ linear phase
$\Rightarrow H_0^2(z) =$ linear phase
$\Rightarrow H_0^2(-z) =$ linear phase
$\Rightarrow T(z) =$ linear phase

For a better understanding: In appendix A is a simple example.

As conclusion the following can be pointed out:

- This linear phase construction is only possible with FIR-filters.
- Just the **minimization** of amplitude distortion is possible.
  This can be achieved eg. with a costfunction $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$
  or more general numerical solutions.

2.2.2 Eliminating Amplitude Distortion

The easiest description of an amplitude distortion free system is that $T(z)$ is a allpass-system. For an exact solution, this is only possible with IIR-Filters.
It is possible to cascade a various number of allpass filters serially (convolution) and the whole system will still behave as a allpass.
Example:

$$\begin{aligned}
A(z) &= \frac{a^* + z^{-1}}{1 + az^{-1}} \cdot \frac{b^* + z^{-1}}{1 + bz^{-1}} && \textit{convolution of 1st order allpasses} \\
A(z) &= \frac{a^*b^* + a^*z^{-1} + b^*z^{-1} + z^{-2}}{1 + az^{-1} + bz^{-1} + abz^{-2}} \\
A(z) &= \frac{(ab)^* + (a+b)^*z^{-1} + z^{-2}}{1 + (a+b)z^{-1} + abz^{-2}} && \textit{2nd order allpass}
\end{aligned}$$

The basic form of the transfer function unfortunately does have a different form:

$$T(z) = \frac{1}{2}[H_0^2(z) - H_0^2(-z)]$$

A shift to the polyphase notation of the basic filter H_0 reveals a slightly different view:

$$\begin{aligned}
T(z) &= \frac{1}{2}[(E_0(z^2) - z^{-1}E_1(z^2))^2 + (E_0((-z)^2) - (-z)^{-1}E_1((-z)^2))^2] \\
&= \frac{1}{2}[(E_0^2(z^2) + 2z^{-1}E_0(z^2)E_1z^2 + z^{-2}E_1^2(z^2)) - (E_0^2(z^2) + 2(-z)^{-1}E_0(z^2)E_1z^2 + z^{-2}E_1^2(z^2))] \\
&= 2 \cdot E_0(z^2)E_1(z^2)
\end{aligned}$$

So $E_0(z)$ and $E_1(z)$ "just" have to be allpass.

The whole system will have the following structure:

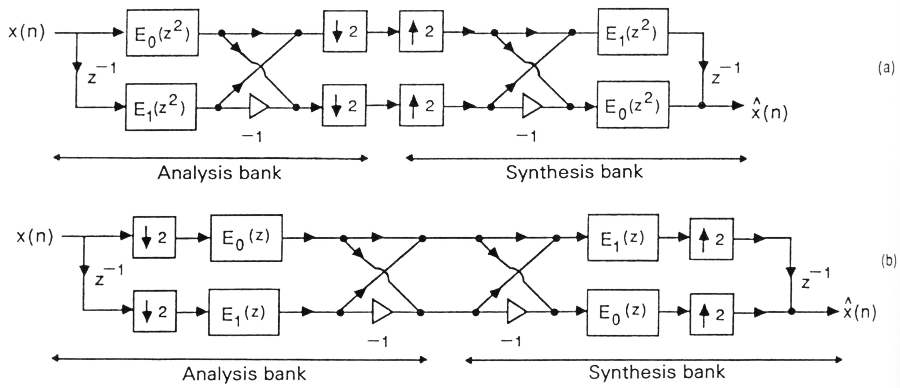

Figure 5.2-2 (a) The complete QMF bank in polyphase form. (b) Rearrangement using noble identities.

Figure 4: QMF in Polyphase-Form [Vaid93]

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1}E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

To be deconstructed into 2 polyphase allpass filters H_0 must fulfill the following conditions:

- The transfer function must be power-symmetric:

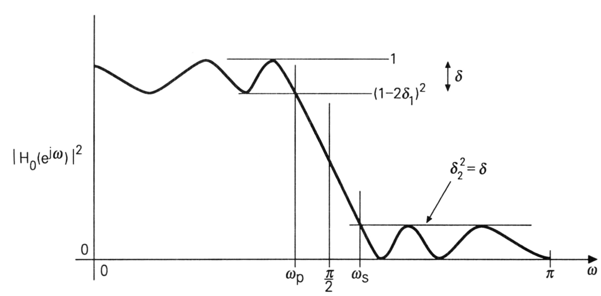

Figure 5.2-6 Square of the magnitude response function for a power symmetric filter.

Figure 5: Power-Symmetric Filter [Vaid93]

- The numerator of the transfer function must be also symmetric

Two "classic" filter types do fulfill these conditions: **Butterworth** and **Elliptic Filters** (Chebycheff filters for example aren't power symmetric). The results are steep and very efficient filters, but with inherent phase distortions.

2.2.3 Perfect Reconstruction Filters

To achieve perfect reconstruction alter the initial conditions for the analysis and synthesis filters.

$$\begin{aligned}
\hat{X}(z) &= \frac{1}{2}[H_0(z)H_1(-z) + H_1(z)H_0(-z)]X(z) \\
H_1(z) &= z^{-N}\tilde{H}_0(-z), \qquad N \geq Order\ of\ H_0,\ N\ odd \\
\tilde{H}_0(z) &= H^{\dagger}(1/z^*), \qquad \text{"}transpose\ conjugate\text{"} \\
\Rightarrow \hat{X}(z) &= -z^{-N}X(z)
\end{aligned}$$

Opposite to the first premise H_1 is not a shifted version of H_0! Of course aliasing is also canceled under these new conditions. From the formulas above further conclusions are obligatory:
H_1 is time reversed compared to H_0 $((1/z^*))$. To remain causal, only FIR-Filters are possible. The delay of the output signal minimally has to be the filter-order. So in practice H_0 is a power-symmetric FIR-filter which can be designed with analytic or numeric FIR design-algorithms.

3 Multi-Channel Filter Banks

3.1 Theory and Basic Transfer Functions

To derive now the conditions for alias cancelation in multi-channel filter banks the procedure is analog to the two-channel case. At first derive the global transfer function, then design the analysis and synthesis filters in a way so that the aliasing term cancels out.

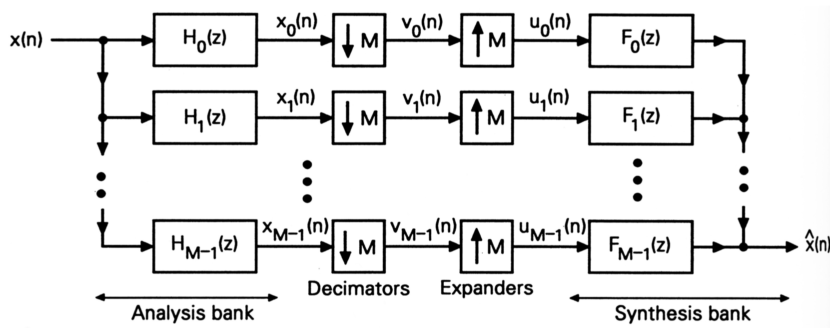

Figure 5.4-1 The M-channel (or M-band) maximally decimated filter bank. Also called M-channel QMF bank.

Figure 6: M-Band QMF-System [Vaid93]

At first the transfer function from X to $\hat{X}$ is calculated.

$$\begin{aligned}
X_k(z) &= H_k(z)X(z) \\
V_k(z) &= \frac{1}{M}\sum_{l=0}^{M-1} H_k(z^{1/M}W^l)X(z^{1/M}W^l), \quad W = e^{-j\frac{2\pi}{M}} \\
U_k(z) &= V_k(z^M) \\
&= \frac{1}{M}\sum_{l=0}^{M-1} H_k(zW^l)X(zW^l) \\
\hat{X}(z) &= \sum_{k=0}^{M-1} F_k(z)U_k(z) \\
&= \frac{1}{M}\sum_{l=0}^{M-1} X(zW^l)\sum_{k=0}^{M-1} H_k(zW^l)F_k(z)
\end{aligned}$$

$$\begin{aligned}
\hat{X}(z) &= \sum_{l=0}^{M-1} A_l(z)X(zW^l), \quad W = e^{-j\frac{2\pi}{M}} \\
A_l(z) &= \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW^l)F_k(z)
\end{aligned}$$

$A_l(z)$ becomes the distortion function $T(z)$, if all aliasing terms are canceled out:

$$\begin{aligned} A_l(z) &= 0 \qquad for\ 1 \le l \le M-1 \\ T(z) &= A_0(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(z)F_k(z) \end{aligned}$$

Or in vector form:

$$M\begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix} = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(zW^{M-1}) & H_1(zW^{M-1})) & \cdots & H_{M-1}(zW^{M-1}) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix} =$$

$$\mathbf{t}(z) = \begin{bmatrix} MT(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left(= \begin{bmatrix} az^{-m_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix} for\ perfect\ reconstruction \right)$$

The task is now to "choose synthesis filters F_k such that overlapping terms cancel out" [Vaid93]. To achieve this, the equation must be rearranged:

$$\begin{aligned} \mathbf{H}(z)\cdot\mathbf{f}(z) &= \mathbf{t}(z) \\ \Rightarrow \mathbf{f}(z) &= \mathbf{H}^{-1}(z)\mathbf{t}(z) \\ \mathbf{f}(z) &= \frac{Adj\mathbf{H}(z)}{det\mathbf{H}(z)}\mathbf{t}(z) \end{aligned}$$

A slight look at this formula reveals some serious problems if this formula should be used to design $\mathbf{f}(z)$. At first is it not unlikely that $\mathbf{f}(z)$ turns out to be IIR even if $\mathbf{H}(z)$ is a FIR-Filter. If all Filters have to be FIR, this is a clear violation of the design specifications. Secondly it is not guaranteed that the matrix containing all analysis filters $\mathbf{H}(z)$ is not singular. In this case, the inversion is not possible because $det\mathbf{H}(z)$ is zero. Even if $\mathbf{H}(z)$ is not singular: the design procedure for $\mathbf{H}(z)$ has to guarantee that its determinant only has zeros within the unit circle. Only then $\mathbf{f}(z)$ is stable - all poles are within the unit circle. All these problems reduce the usability of this "pure" form severely.

One way to overcome these obstacles is simply to remove the determinant from the formula. In this case amplitude and phase distortions are immanent, the design goals are reduced to alias cancelation. The new formula is then $\mathbf{f}(z) = [Adj\mathbf{H}(z)]\cdot\mathbf{t}(z)$

In practice this downgraded method also implies some constraints. $\mathbf{f}(z)$ is definitely FIR but can be of very large order. Another drawback is the possibility of drastic amplitude distortions contained in the new distortion transfer function $\mathbf{t}(z) = cz^{-m_0}\cdot[det\mathbf{H}(z)]$. The reason for this is that the determinant of $H(z)$ could have zeros on or very near the unit circle.

3.2 Polyphase Representation

A solution for these problems is possible shifting to the polyphase representation of the system.

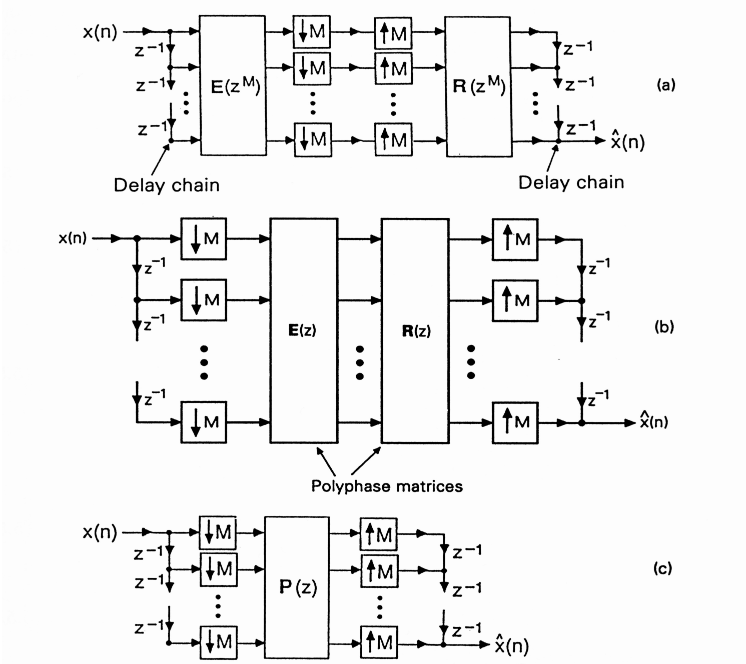

Figure 5.5-3 (a) Polyphase representation of an M-channel maximally decimated filter bank. (b) Rearrangement using noble identitites. (c) Further simplification, where $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$.

Figure 7: Polyphase Representation [Vaid93]

$$\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$$

The analysis section is a Type 1 Polyphase Form:

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M)$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0(M-1)}(z^M) \\ E_{10}(z^M) & E_{11}(z^M) & \cdots & E_{1(M-1)}(z^M) \\ \vdots & \vdots & \ddots & \vdots \\ E_{(M-1)0}(z^M) & E_{(M-1)1}(z^M) & \cdots & E_{(M-1)(M-1)}(z^M) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

$$\mathbf{h}(z) = \mathbf{E}(z^M)\mathbf{e}(z)$$

The synthesis section is a Type 2 Polyphase Form:

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{kl}(z^M)$$

$$\begin{bmatrix} F_0(z) & F_1(z) & \cdots & F_{M-1}(z) \end{bmatrix} =$$

$$
\begin{bmatrix} z^{-(M-1)} & z^{-(M-2)} & \cdots & 1 \end{bmatrix}
\begin{bmatrix}
R_{00}(z^M) & R_{01}(z^M) & \cdots & R_{0(M-1)}(z^M) \\
R_{10}(z^M) & R_{11}(z^M) & \cdots & R_{1(M-1)}(z^M) \\
\vdots & \vdots & \ddots & \vdots \\
R_{(M-1)0}(z^M) & R_{(M-1)1}(z^M) & \cdots & R_{(M-1)(M-1)}(z^M)
\end{bmatrix}
$$

$$
\mathbf{f}^T(z) = (z^{-(M-1)})\tilde{\mathbf{e}}(z)\mathbf{R}(z^M)
$$

Because of the noble identities it is possible to place the polyphase filters after the downsampling stage (analysis filter $E(z)$) respectively before the upsampling stage (synthesis filter $R(z)$). As the two Matrices are no longer divided by the downsampling/upsampling stages they can be merged into one big matrix. Now it is possible to design a filter matrix $P(z)$ which satisfies the various design specifications regarding aliasing, amplitude and phase distortions. From the later derived criteria for $P(z)$ we are able to extract design criteria for $E(z)$ and $R(z)$ which contain much more degrees of freedom than in the "normal" form from section 3.1. A big practical preface is also that the polyphase form of the filters can be implemented directly to a dsp and is quite effective regarding the processor load.

3.3 Alias Free Systems

The first step is again to remove the aliasing. In this paper the target is to derive the mathematical conditions for $P(z)$ to omit aliasing. Once this is done, known design principles can be checked concerning their compatibility. So again the first step is to derive the transfer function. This time it contains $\mathbf{P}(z)$.

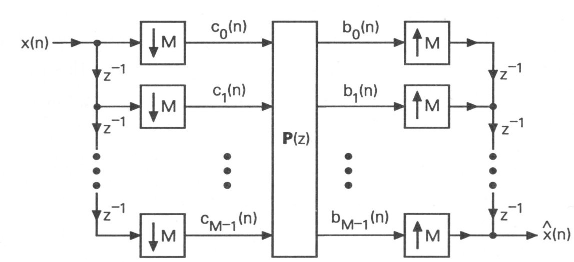

Figure 5.7-3 The equivalent circuit for the maximally decimated filter bank.

Figure 8: Polyphase Representation 2 [Vaid93]

$$
\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)
$$

$$\begin{aligned}
\hat{X}(z) &= \frac{1}{M}\sum_{s=0}^{M-1} z^{-(M-1-s)} B_s(z^M) \\
\hat{X}(z) &= \frac{1}{M}\sum_{s=0}^{M-1} z^{-(M-1-s)} \sum_{l=0}^{M-1} P_{s,l}(z^M) C_l(z^M) \\
\hat{X}(z) &= \frac{1}{M}\sum_{s=0}^{M-1} z^{-(M-1-s)} \sum_{l=0}^{M-1} P_{s,l}(z^M) \sum_{k=0}^{M-1} (zW^k)^{-l} X(zW^k)
\end{aligned}$$

Finally it is possible to change the order of the formula to prepare it for further considerations. The auxiliary variable V_l is introduced just to simplify our further investigations.

$$\begin{aligned}
\hat{X}(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X(zW^k) \sum_{l=0}^{M-1} W^{-kl} \sum_{s=0}^{M-1} z^{-l} z^{-(M-1-s)} P_{s,l}(z^M) \\
\hat{X}(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X(zW^k) \sum_{l=0}^{M-1} W^{-kl} V_l \\
V_l &= \sum_{s=0}^{M-1} z^{-l} z^{-(M-1-s) P_{s,l}(z^{-M})}
\end{aligned}$$

Analog to the prior solutions it is possible to determine an alias cancelation condition:

$$\sum_{l=0}^{M-1} W^{-kl} V_l = 0 \quad for\ all\ k \neq 0$$

In the more comprehensible vector form:

$$\mathbf{W}^{\dagger} \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \Omega \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \mathbf{W} \begin{bmatrix} \Omega \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

W is the DFT-Matrix. A description of it can be found in appendix B.
The exact content of Ω is of no importance at this moment. It can be arbitrary. It just influences the distortion transfer function, which can be arbitrary at this point. With one exception: the distortion transfer function and hence also Ω must not be zero.

To remain comprehensible the further explanations will be made in a system with $M = 3$. All conclusions are of course also valid for a system of arbitrary size. At the end all "complicated" reflections will lead again to a simple general rule concerning the structure of **P**.
For $M = 3$ the system looks like this:

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}2} \\ 1 & e^{-j\frac{2\pi}{3}2} & e^{-j\frac{2\pi}{3}4} \end{bmatrix} \begin{bmatrix} \Omega \\ 0 \\ 0 \end{bmatrix}$$

To satisfy this condition all $V_l(z)$ have to be Ω and thus the one and only $V(z)$.

$$\Rightarrow V_0(z) = V_1(z) = V_2(z) = \Omega = V(z)$$

Or more generally:

$$\Rightarrow V_l(z) = V(z) \quad for\ 0 \leq l \leq M-1$$

A closer look at what elements are contained in the particular V_l will demonstrate what conditions the singular elements of $\mathbf{P}$ have to fulfill.

$$\begin{array}{rcrcr}
\mathbf{V}_0(\mathbf{z}) & & \mathbf{V}_1(\mathbf{z}) & & \mathbf{V}_2(\mathbf{z}) \\
 & & & & z^{-4}P_{0,2}(z^3) \\
 & & z^{-3}P_{0,1}(z^3) & & +z^{-3}P_{1,2}(z^3) \\
z^{-2}P_{0,0}(z^3) & = & +z^{-2}P_{1,1}(z^3) & = & +z^{-2}P_{2,2}(z^3) \\
+z^{-1}P_{1,0}(z^3) & & +z^{-1}P_{2,1}(z^3) & & \\
+z^{0}\ P_{2,0}(z^3) & & & &
\end{array}$$

All V_l are only equal in case the elements $P_{s,l}$ have the following relationships:

$$\begin{aligned}
P_{0,0}(z^3) &= P_{1,1}(z^3) = P_{2,2}(z^3) = P_0(z^3) \\
P_{1,0}(z^3) &= P_{2,1}(z^3) = z^{-3}P_{0,2}(z^3) = z^{-3}P_2(z^3) \\
P_{2,0}(z^3) &= z^{-3}P_{0,1}(z^3) = z^{-3}P_{1,2}(z^3) = z^{-3}P_1(z^3)
\end{aligned}$$

So the matrix $\mathbf{P}$ has to look exactly like this to prevent aliasing:

$$\mathbf{P}(z) = \mathbf{E}(z)\mathbf{R}(z) = \begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix}$$

Most generally spoken:
Every alias free system in polyphase form must contain a **pseudocirculant** P-matrix.
A pseudocirculant matrix is a matrix, where every row is a right-shifted copy of the row before with the small extension, that all elements under the main diagonal possess an additional z^{-1}.

3.4 Perfect Reconstruction Filters

Regarding this very general and powerful condition, a countless number of alias free systems realizations are possible. For further investigations the interested reader reader may have a look at [Vaid93] or similar sources. This last chapter contains a brief view into the most general structure of a perfect reconstruction filter. The not unusual design criteria are the following:

- All filters should be FIR (polyphase decomposition is simple, easy linear phase implementation).
- M can be arbitrary.
- $H_k(z)$ provides as much attenuation as the user specifies.
- The implementation cost should be competitive with approximate reconstruction systems.

After a brief look at figure 8, a very intuitive solution that also inherits the non-aliasing condition seems quite obvious:

$$\mathbf{P}(z) = c \cdot z^{-m_0}\mathbf{I}, \quad m_0 \geq M \; for\ causality$$

Actually the most general condition all perfect reconstruction systems have to meet:

$$\mathbf{P}(z) = c \cdot z^{-m_0} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(M-r)\times(M-r)} \\ z^{-1}\mathbf{I}_{r\times r} & \mathbf{0} \end{bmatrix}, \quad 0 \leq r \leq M-1, c \neq 0$$

$$T(z) = cz^{-r}z^{-(M-1)}z^{m_0 M}$$

Looking at the alias-free condition it's clear, that this formula is within the expectations. But perfect reconstruction? The easiest system possible will clear things up.
M is 3, $P_{s,l}$ is either 1 or 0. Thus there is no "real" filtering done in this systems. The samples are divided at the entry of the system and put together again at the end. The most complex component is a delay.

$$\mathbf{P}(z) = \begin{bmatrix} 0 & 0 & 1 \\ z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \end{bmatrix}$$

$$\mathbf{b}(z) = \mathbf{P}(z)\mathbf{c}(z)$$

Figure 9 sketches the implemented system and shows the contents of the system parts at each position. As predicted, the samples are leaving the system in correct order. Just a delay is inserted. It amounts to

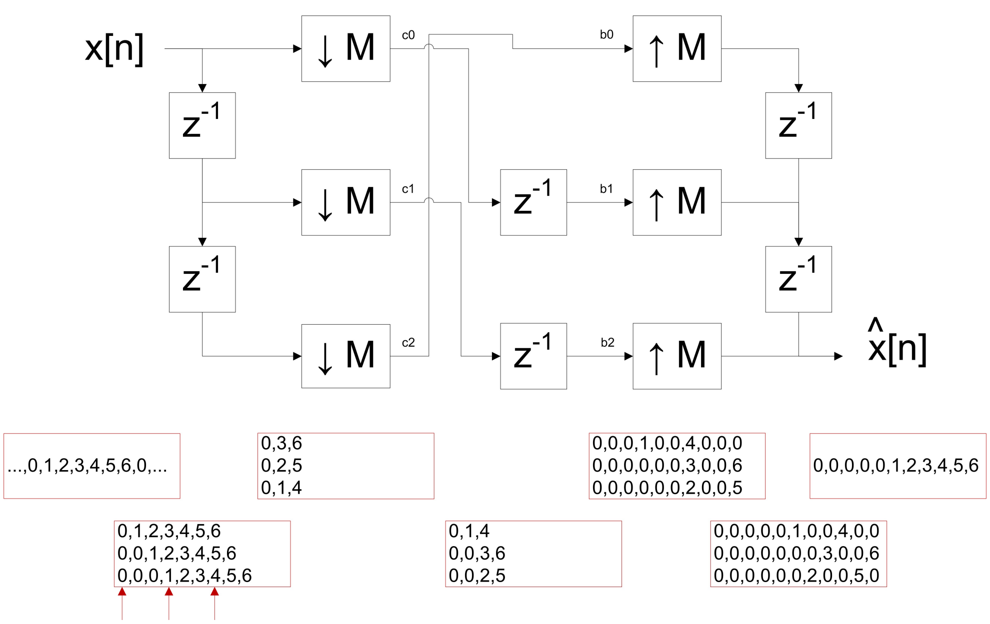

Figure 9: Example Perfect Reconstruction

$z^{-r-M+1} = z^{-4}$.

4 Summary

In the preceding sections the mathematic theory and some examples of Maximally Decimated Filter Banks has been discussed. Hopefully the reader has gained a basic understanding and knowledge. Sources for further investigations are nearly as countless as the need for filter banks in modern DSP systems. The IEEE Explorer used on May 14th 2007 exhibited 446532 documents concerning "Maximally Decimated Filter Bank" and 1566306 documents concerning "QMF Bank". As this is hardly manageable to read in a lifetime, the author would be delighted, if this essay proofed useful as a shortcut and starting point for the users research in this wide field.

Christian Goettlinger, July 7. 2007

A Example: Eliminating Phase Distortion

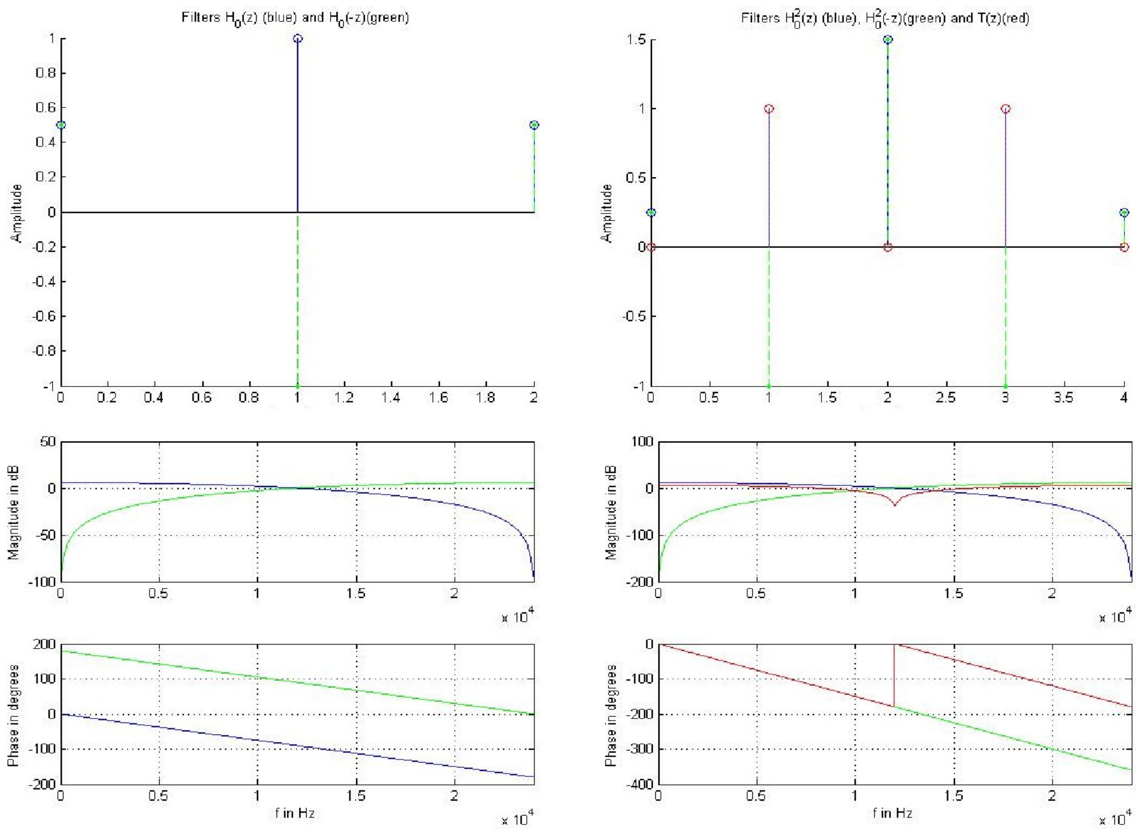

Figure 10: Phase Distortion Example

In our example we used a very simple linear phase FIR as a prototype for H_0:

$$\begin{aligned}
H_0(z) &= 0.5 + z^{-1} + 0.5z^{-2} && \textit{linear phase lowpass filter} \\
\Rightarrow H_0(-z) &= 0.5 - z^{-1} + 0.5z^{-2} && \textit{linear phase highpass filter}
\end{aligned}$$

$$\begin{aligned}
H_0^2(z) &= (0.5 + z^{-1} + 0.5z^{-2})^2 \\
&= 0.25 + z^{-1} + 1.5z^{-2} + z^{-3} + 0.25z^{-4} && \textit{linear phase lowpass filter} \\
H_0^2(-z) &= (0.5 - z^{-1} + 0.5z^{-2})^2 \\
&= 0.25 - z^{-1} + 1.5z^{-2} - z^{-3} + 0.25z^{-4} && \textit{linear phase highpass filter}
\end{aligned}$$

$$\begin{aligned}
T(z) &= 1/2[H_0^2(z)] - H_0^2(-z)] \\
&= 1/2[(0.25 + z^{-1} + 1.5z^{-2} + z^{-3} + 0.25z^{-4}) \\
&\qquad -(0.25 - z^{-1} + 1.5z^{-2} - z^{-3} + 0.25z^{-4})] \\
&= z^{-1} + z^{-3} && \textit{linear phase filter}
\end{aligned}$$

B The DFT Matrix

The DFT-Matrix has the following appearance:

$$\begin{aligned} \mathbf{W} &= \begin{bmatrix} W^{00} & W^{01} & \cdots & W^{0(N-1)} \\ W^{10} & W^{11} & \cdots & W^{1(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W^{(N-1)0} & W^{(N-1)1} & \cdots & W^{(N-1)(N-1)} \end{bmatrix} \\ W &= e^{-j\frac{2\pi}{N}} \end{aligned}$$

$$\Rightarrow \mathbf{W} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N\cdot 1}} & \cdots & e^{-j\frac{2\pi}{N}\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{N\cdot(N-1)}} & \cdots & e^{-j\frac{2\pi}{N}\cdot(N-1)^2} \end{bmatrix}$$

The Matrix has some very useful special properties:

$$\begin{aligned} \mathbf{W}^T &= \mathbf{W} && \textit{symmetric matrix} \\ \mathbf{W}^H &= \mathbf{W}^* \\ \mathbf{W}^H \cdot \mathbf{W} &= N \cdot \mathbf{I} && \text{"}\textit{easy}\text{" } \textit{inversion} \\ \mathbf{W}^{-1} &= \frac{\mathbf{W}^*}{N} && \textit{unitary matrix} \end{aligned}$$

References

[Vaid93] P.P. Vaidyanathan *Multirate Systems And Filter Banks.* Prentice Hall, 1993.

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[IEEE07] IEEE Explorer, 05.14.2007