Signaltransformationen SPSC

1a) \[ x[n] = \sum_{k=0}^{N_o-1} a_k \cdot e^{j \frac{2\pi}{N_o} kn} = \sum_{k=0}^{3} e^{j \frac{2\pi}{N} kn} \]
\[ = 1 + e^{j \frac{2\pi}{N} n} + (-1)^n + e^{-j \frac{2\pi}{N} n} \]
\[ = 1 + (-1)^n + 2 \cos\left(\frac{\pi}{N} n\right) \]
\[ x[0] = 2 + 2 \cdot 1 = 4 \]
\[ x[2] = 2 - 2 \cdot (-1) = 0 \]
\[ x[4] = 0 + 2 \cdot 0 = 0 \]
\[ x[3] = 0 + 2 \cdot 0 = 0 \]

1a alternative)
\[ a_k = 1 = e^0 = \frac{1}{N_o} \cdot \sum_{n=0}^{N_o-1} x[n] \cdot e^{j \frac{2\pi}{N_o} kn} = \frac{1}{4} \cdot \sum_{n=0}^{3} x[n] \cdot e^{j \frac{2\pi}{N} kn} \]
\[ e^0 = \frac{1}{4} \left( x[0] \cdot e^0 + x[2] \cdot e^{j \frac{2\pi}{N} k} + x[2] \cdot e^{j \frac{2\pi}{N} k} + x[3] \cdot e^{j \frac{2\pi}{N} k} \right) \]
Koeffizientenvektor: \[ = 4 \]
\[ = 0 \]
\[ = 0 \]
\[ = 0 \]

1b) \[ x[n] \]
\[ \begin{array}{c}
4 \quad 3 \quad 2 \quad 1 \\
\hline
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \\
\end{array} \]
2 a) \[ x(t) = \cos\left(\frac{\pi}{2}t\right) + \cos\left(\frac{\pi}{3}t\right) \]

\[ \omega_1 = \frac{\pi}{2} \Rightarrow T_1 = \frac{2\pi}{\omega_1} = 4\pi \quad \omega_2 = \frac{\pi}{3} \Rightarrow T_2 = \frac{2\pi}{\omega_2} = 6\pi \]

\[ T_0 = \log V(4\pi, 6\pi) = 2\pi \cdot \log V(2, 3) = 12\pi \]

\[ \omega_0 = \frac{\omega_1}{T_0} = \frac{\omega_2}{T_0} = \frac{1}{6} \checkmark \]

2 b) \[ x(t) = \cos(3 \cdot \frac{4}{6}t) + \cos(2 \cdot \frac{4}{6}t) = \cos(3\omega_0 t + \cos(2\omega_0 t) \]

\[ = \frac{1}{2} e^{3\omega_0 t} + \frac{1}{2} e^{-3\omega_0 t} + \frac{1}{2} e^{2\omega_0 t} + \frac{1}{2} e^{-2\omega_0 t} \]

\[ = \sum_{k \in \mathbb{Z}} a_k e^{i k\omega_0 t} \checkmark \]

Koeffizientenvergleich:

\[ a_k = \left\{ \begin{array}{ll} \frac{1}{4} & k \in \{-3, -2, 2, 3\} \\ 0 & \text{sonst} \end{array} \right\} \checkmark \]
3a) $|X(j\omega)|$  

\[ \begin{align*} 
\text{3b über Inv. FT) } & \text{ also } X(j\omega) = |X(j\omega)| \cdot e^{\jmath \text{arg } X(j\omega)} \text{ follows.} \\
X(j\omega) &= \begin{cases} 
2 \cdot e^{-j\omega} & \text{when } -3 \leq \omega \leq 3 \\
0 & \text{otherwise}
\end{cases} \\
\text{Hence,} \\
x(t) &= \frac{1}{\jmath \pi} \cdot \int_{\mathbb{R}} X(j\omega) e^{j\omega t} \, d\omega \\
&= \frac{1}{\jmath \pi} \cdot 2 \int_{-3}^{3} e^{-j\omega} \, e^{j\omega t} \, d\omega \\
&= \frac{2}{\jmath \pi} \cdot \frac{1}{\jmath \pi} \left( e^{j3(t-1)} - e^{-j3(t-1)} \right) \\
&= \frac{2}{\jmath \pi} \cdot \frac{\sin(3(t-1))}{t-1} \\
&\in \mathbb{R}
\end{align*} \]

3b über Zeitverschiebungsregel

Definition Rechteckfunktion: 

\[ \text{rect}(\omega) := \begin{cases} 
1 & \text{when } -\frac{1}{2} \leq \omega \leq \frac{1}{2} \\
0 & \text{otherwise}
\end{cases} \]

Mit $a > 0$ gilt: 

\[ \text{rect}(\frac{\omega}{a}) \rightarrow \frac{\sin(\pi t)}{\pi t} \]

Damit ergibt sich 

\[ X(j\omega) = 2 \cdot \text{rect}(\frac{\omega}{3}) \cdot e^{-j\omega} \]

(siehe oben)

\[ \text{rect}(\frac{\omega}{3}) = \text{rect}(\frac{\omega}{2 \cdot \frac{3}{2}}) \rightarrow \frac{\sin(3t)}{3 \pi t} \]

(konstanter Faktor)

\[ 2 \cdot \text{rect}(\frac{\omega}{2 \cdot \frac{3}{2}}) \rightarrow 2 \cdot \frac{\sin(3t)}{3 \pi t} \]

(zeitverschiebungsregel)

\[ X(j\omega) = 2 \cdot \text{rect}(\frac{\omega}{2 \cdot \frac{3}{2}}) \cdot e^{-j\omega} \rightarrow 2 \cdot \frac{\sin(3(t-1))}{\pi (t-1)} = x(t) \]