Homework Assignment 5

Name(s) **Matr.No(s)**

Your homework has to be delivered as hard copy at the beginning of the lecture on **2011/5/11**. Use a printed version of this assignment document as the title page(s) and fill in your name(s) and matriculation number(s). You may work in pairs (only one copy needs to be delivered per pair).

Analytical Problem 5.1 (10 Points)

Consider a random variable X with a density $f_X(x)$ and the absolute-error distortion criterion $d(x, \hat{x}) = |x - \hat{x}|$. We want to quantize X using a scalar quantizer based on a high-rate design with step sizes Δ_i and the corresponding centroid density $g_C(x)$.

(a) Derive the relation between the mean distortion of a single cell D_i and Δ_i.

(b) Derive an expression for the overall mean distortion as a function of the centroid density $g_C(x)$.

(c) Under the constraint that the total number of centroids is N, derive the optimal centroid density function $g_C(x)$.

(d) Show that each quantization cell contributes the same amount to the overall mean distortion.

(e) For this and the following tasks, assume X to have a Laplacian distribution $f_X(x) = \frac{a}{2}e^{-a|x|}$. Find the relation between rate and distortion for the constrained-resolution scalar quantizer.

(f) Find the relation between rate and distortion for the constrained-entropy scalar quantizer.

(g) Sketch the rate-distortion relations obtained in the two previous tasks and compare them to the rate-distortion function $R^*(D)$. (You may copy the results from examples of Chapter 6.)

Analytical Problem 5.2 (2 Points)—Bonus Problem 1

Consider the jointly Gaussian (vector) variable $\mathbf{X}$ with zero mean and the covariance matrix

$$C_{\mathbf{X}} = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}.$$

We want to design an entropy-constrained, 2-dimensional VQ for i.i.d. samples of $\mathbf{X}$. We assume a rate sufficiently high for the common high-rate approximations.

(a) What is the optimal cell shape for a 2-dimensional VQ?

(b) For a given value R for the entropy of the quantization index, calculate the optimal lattice generator matrix G^{T}. (Hint: consider G^{T} as $G^{\mathsf{T}} = \alpha G_1^{\mathsf{T}}$, where α is a positive scalar and G_1^{T} is some normalized generator matrix for optimal cell shapes. For instance, normalization may be with respect to the cell area or with respect to the length of the grid vectors.)

(c) For the above designed lattice VQ, compute the mean distortion (mean squared reconstruction error) D per dimension.

MATLAB Problem 5.3 (5 Points)—Bonus Problem 2

(a) Generate 100 000 realizations of the random vector $\mathbf{X}$ defined above in the previous problem. (Hint: consider $\mathbf{X}$ as $\mathbf{X} = B\widetilde{\mathbf{X}}$ where the components of $\widetilde{\mathbf{X}}$ are independent Gaussians.) These samples will be used as the test data to evaluate the performance of the VQ. Make a scatter plot.

(b) Write a program that designs a lattice VQ (i.e., its generator matrix) for a given entropy ('design rate') as done in the analytical problem above. The program should also compute the high-rate approximation of the mean distortion D ('design distortion'). For $R = 4$ bits, plot the centroids on top of the previously made scatter plot. To facilitate a later implementation, draw some neighboring cell boundaries in the plot (this may be done by hand).

(c) Write the function `[Xq, U] = lattice_vq( X, GT)` that implements the lattice VQ using a fast-search procedure[1]. `X` are vector samples (one or many), `GT` is the generator matrix, `Xq` are the nearest-neighbor centroids for `X`, and `U` are the corresponding, 2-dimensional cell indices. (Note, for our simulation we do not need to implement any lattice truncation, since we will not implement any lossless coder.)

(d) Evaluate the rate-distortion performance of the lattice VQ for design rates between 1 and 6 bits (in 0.5 bit steps) using the test data. We can expect that the actual performance deviates from the design performance (particulary at lower rates). For the actual rate, we estimate the asymptotic rate (with block length $\to \infty$) of a lossless coder that approximates the cell probabilities as $p_I(i) = V_i \cdot \mathcal{N}(c_i|C_{\mathbf{X}})$ where V_i is the cell volume (area) and c_i is the centroid. Plot the design performance (design rate over design distortion) as well as the actual performance (actual rate over actual distortion) in one graph.

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[1] How many cells need to be considered for the 2-dimensional case?