Fundamentals of Digital Communications
Class 2: Geometric Representation of Signal Waveforms

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SCHEDULE – to master Digital Communications

Demodulation and Detection Theory

Class 7
Class 8

Signal Spaces

Class 1
Class 2

Signals and Systems

Class 3
Class 4

Stochastic

Class 5
Class 6
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Demodulation and Detection Theory

Class 8

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Class 5
Goals:

- understand Gram-Schmidt-orthogonalization
- understand subspaces and the projection onto them
- understand signal and vector spaces (a little geometry)
- compute the distance between two signal waveforms
**Problem 2.1**

**Given:**

\[ s_1(t) \]

-1 \[ \quad \] 1 \[ \quad \] 2 \[ \quad \] 3 \[ \quad \] 4 \[ \quad \] t

\[ s_2(t) \]

-2 \[ \quad \] -1 \[ \quad \] 0 \[ \quad \] 1 \[ \quad \] 2 \[ \quad \] t

\[ s_3(t) \]

-2 \[ \quad \] -1 \[ \quad \] 0 \[ \quad \] 1 \[ \quad \] 2 \[ \quad \] t

\[ s_4(t) \]

-2 \[ \quad \] -1 \[ \quad \] 0 \[ \quad \] 1 \[ \quad \] 2 \[ \quad \] t
Problem 2.1 (cont’d)

Question:

- find coefficient vectors $p_i$ in 4-D space $\mathcal{P} = \text{span} \{\phi_1(t), \ldots, \phi_2(t)\}$
- use Gram-Schmidt orthogonalization to find dimensionality $N$ and basis $q_j$, $j = 1, \ldots, N$ for $\mathcal{S} = \text{span} \{p_1, \ldots, p_4\}$
- sketch corresponding orthonormal basis functions $\{\psi_j(t)\}_{j=1}^N$ for $\mathcal{S} = \text{span} \{s_1(t), \ldots, s_4(t)\}$
- determine $s_i$ for signal space $\mathcal{S}$
- distance between signals $s_1(t)$ and $s_2(t)$:
  1. directly,
  2. from their vectorial representation in $\mathcal{P}$, and
  3. from their vectorial representation in $\mathcal{S}$. 
**Problem 2.2**

Given:

\[ u_1(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_1 t) \quad 0 \leq t \leq T \]

\[ u_2(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi f_2 t) \quad 0 \leq t \leq T \]

Question:

- Find and maximize the distance between the waveforms

\[ d = \|u_2(t) - u_1(t)\| \]

as a function of the frequency spacing

\[ \Delta f = f_2 - f_1. \] Assume \( f_1, f_2 \gg \frac{1}{T} \)
Problem 2.2

Sinus Cardinalis

\[ \text{sinc}(2\Delta f) \]

\( X: 0.715 \)
\( Y: -0.2172 \)
Questions

- How can you determine the dimension of a signal space?
- What does the projection theorem state?
- Explain the Gram-Schmidt procedure?
QUESTIONS

- How can you determine the dimension of a signal space? via Gram-Schmidt
- What does the projection theorem state? projection error is orthogonal to subspace
- Explain the Gram-Schmidt procedure?
  1. Choose one vector, normalize it → first basis.
  2. Project next vector onto current basis. If projection error is zero, no new basis. Otherwise normalize projection error (orthogonal to current basis) → next basis.
  3. Repeat step 2 for all remaining vectors.
Orthonormal Basis (cont’d)

Gram-Schmidt orthogonalization

- find the $N \leq M$ orthonormal basis functions

  - $\psi_1(t) = \frac{s_1(t)}{\|s_1(t)\|}$
    
    $S_1 = \text{span} \{ \psi_1(t) \}$

  - $\hat{s}_2(t) = c_{21} \psi_1(t)$
    
    $\psi_2(t) = \frac{s_2(t) - \hat{s}_2(t)}{\|s_2(t) - \hat{s}_2(t)\|}$
    
    $S_2 = \text{span} \{ \psi_1(t), \psi_2(t) \}$

  - projection of $s_2(t)$ onto $S_1$

  - $\vdots$

  - $\psi_k(t) = \frac{s_k(t) - \hat{s}_k(t)}{\|s_k(t) - \hat{s}_k(t)\|}$
    
    $\hat{s}_k(t) = \sum_{i=1}^{k-1} c_{ki} \psi_i(t)$
    
    $S_k = \text{span} \{ \psi_1(t), \psi_2(t), \ldots, \psi_k(t) \}$

  - projection of $s_k(t)$ onto $S_{k-1}$

  - no basis function, if $s_k(t) - \hat{s}_k(t) = 0$