# Supplement: Random Variables and Lebesgue Decomposition

Bernhard Geiger

Graz University of Technology

Winter Term 2017/18

# Review

- Probability Space
- Random Variables
- Three Types of Distributions
- Lebesgue Decomposition

# Probability Space

- Sample space $\Omega$
- $\sigma$-algebra $\mathcal{F}$
- Probability measure $\mathbb{P}$: $\mathcal{F} \to [0, 1]$

# Random Variable

- It's a function (measurable): $X\colon \Omega \to \mathbb{R}^N$
- We call its alphabet $\mathcal{X} \subseteq \mathbb{R}^N$
- It induces a new probability space $(\mathcal{X}, \mathfrak{B}(\mathcal{X}), P_X)$ where

$$\forall A \in \mathfrak{B}(\mathcal{X}): \quad P_X(A) = \mathbb{P}(X^{-1}(A))$$

# Volume (Lebesgue Measure)

$\lambda^N(A)$ is the *volume* of $A \subset \mathbb{R}^N$

Intuition:

- $\lambda^1$ is length
  - $\lambda^1([a,b]) = b - a$
- $\lambda^2$ is area
  - $\lambda^2([a,b] \times [c,d]) = (b-a)(d-c)$
  - $\lambda^2([a,b] \times \{y\}) = 0$
- $\lambda^3$ is 3D-volume
- ...

If $A \subset \mathbb{R}^N$ is countable, then $\lambda^N(A) = 0$.

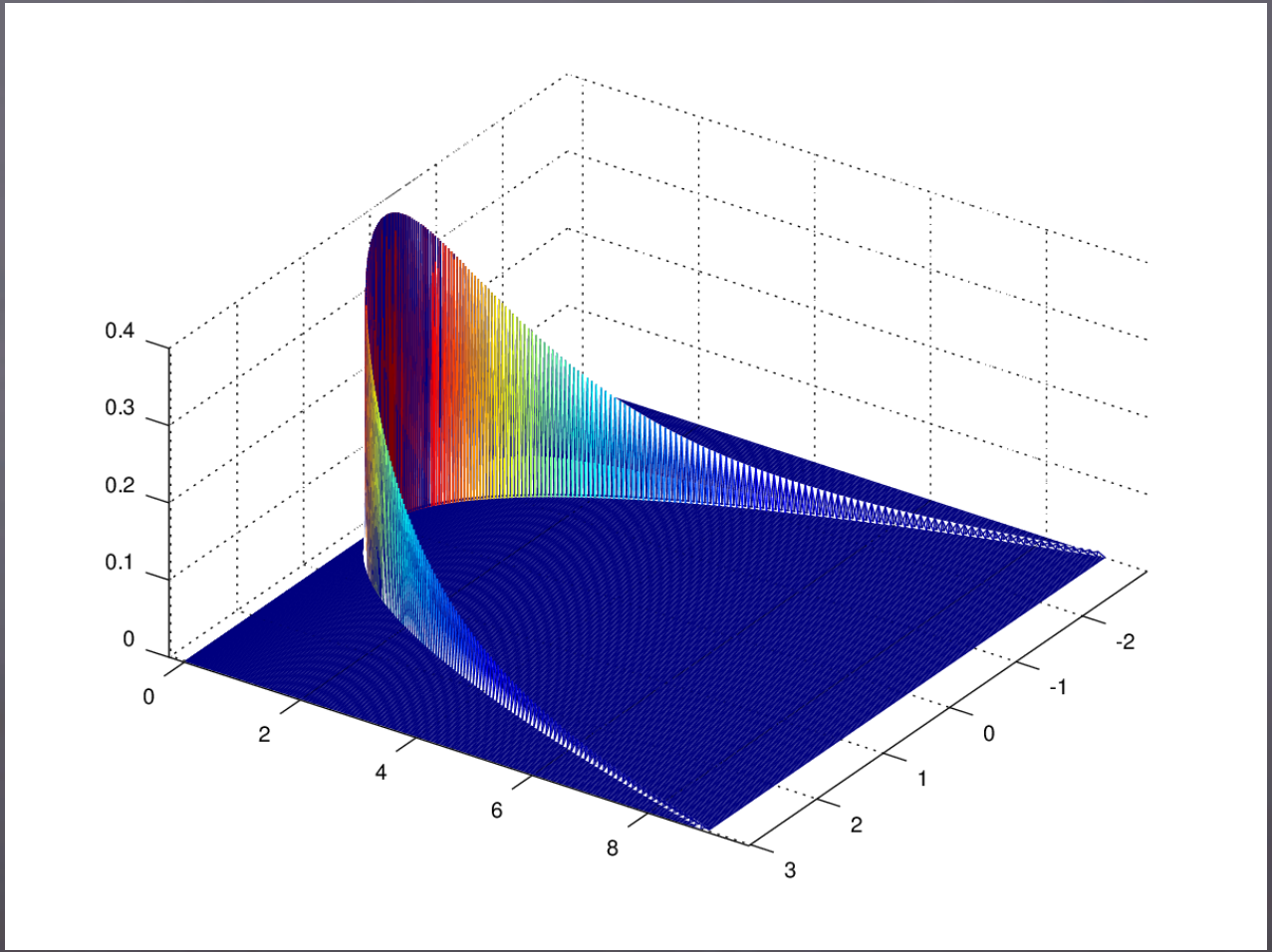
# Three Types of Distributions

Support $\mathcal{X} \subseteq \mathbb{R}^N$, $x \in \mathcal{X}$

| Type | $\lambda^N(\mathcal{X})$ | $P_X(\{x\}) > 0$ | $H(X)$ | $h(X)$ |
|---|---|---|---|---|
| Disc. $P_X^d$ | $= 0$ | $> 0$ | OK | $-\infty$ |
| Cont. $P_X^{ac}$ | $> 0$ | $= 0$ | $\infty$ | OK |
| (D-C-Mixt. | $> 0$ | $> 0$ | $\infty$ | ?) |
| Sing. $P_X^{sc}$ | $= 0$ | $= 0$ | ? | ? |

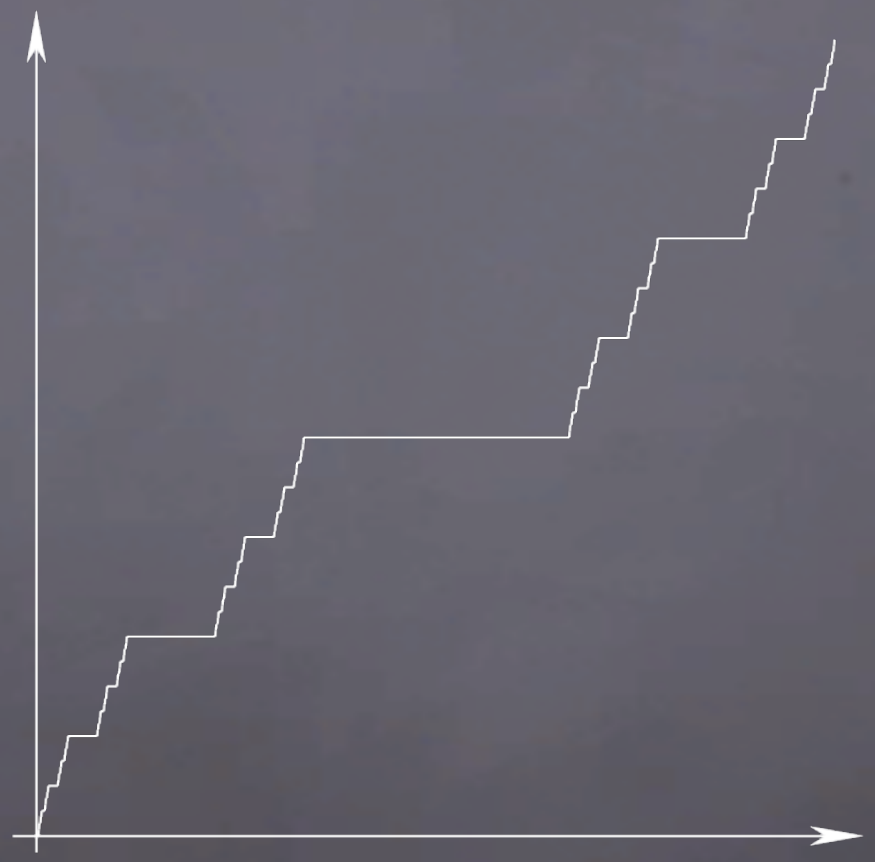
# Joint Distribution Between Input and Output

$X$ is zero-mean, unit variance Gaussian, $Y = X^2$.



## Cantor-Distribution

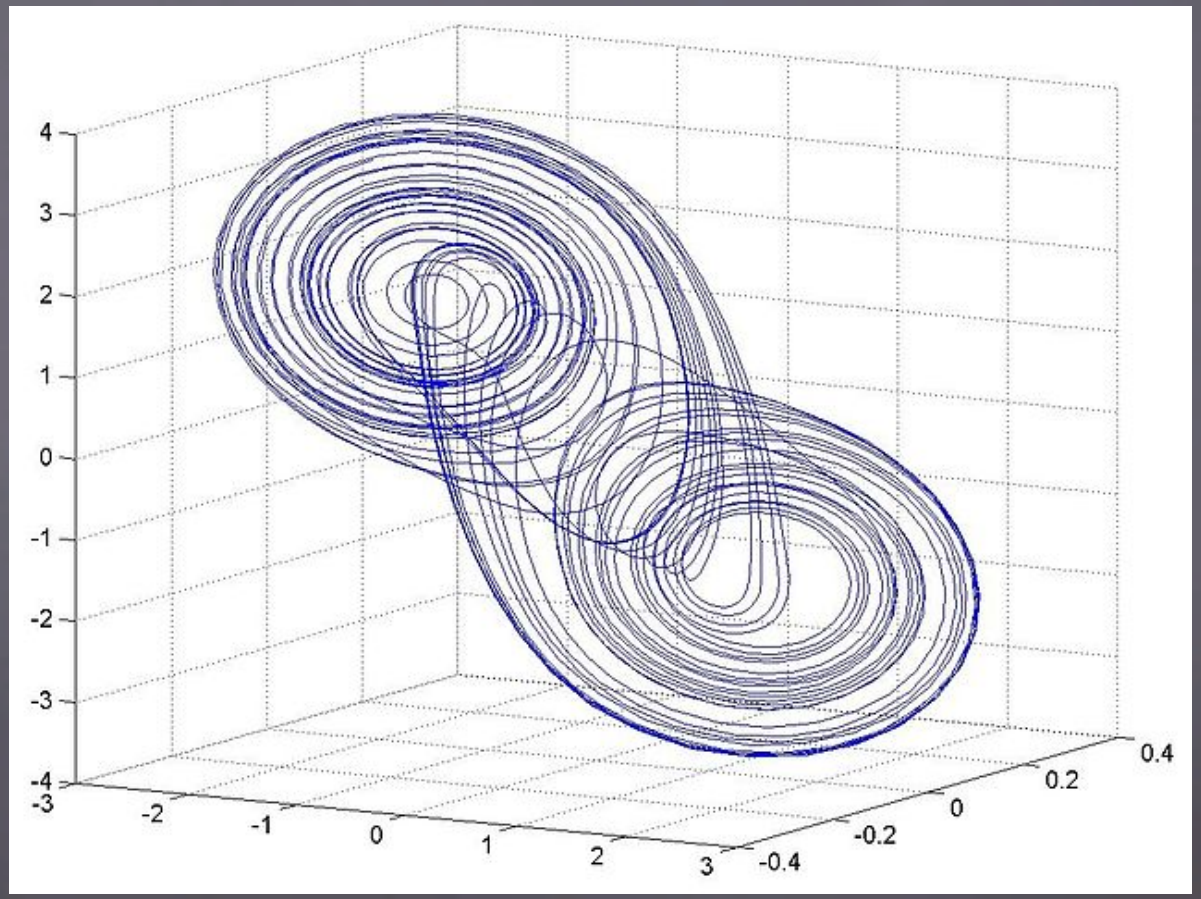
"Uniform distribution on the Cantor set $\mathcal{C}$"[1]; $\lambda(\mathcal{C}) = 0$. The CDF is continuous, but its derivative is zero a.e.



[1](CC-BY-SA-3.0 Theon)

# Attractor of Chua's Circuit

"Double-Scroll Attractor" of Chua's Circuit[2]




[2](CC-BY-SA-3.0 www.chuacircuits.com)

# Lebesgue-Decomposition

Let $X$ be a RV with probability distribution $P_X$, where

$$P_X(A) = \mathbb{P}(X \in A)$$

Every $P_X$ can be decomposed into the three types from the table:

$$\forall A \subseteq \mathcal{X}: \quad P_X(A) = P_X^d(A) + P_X^{ac}(A) + P_X^{sc}(A)$$