Supplement:
Random Variables and Lebesgue Decomposition

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Review

- Probability Space
- Random Variables
- Three Types of Distributions
- Lebesgue Decomposition
Probability Space

- Sample space $\Omega$
- $\sigma$-algebra $\mathcal{F}$
- Probability measure $\mathbb{P}: \mathcal{F} \rightarrow [0, 1]$
Random Variable

- It's a function (measurable): \(X: \Omega \to \mathbb{R}^N\)
- We call its alphabet \(\mathcal{X} \subseteq \mathbb{R}^N\)
- It induces a new probability space \((\mathcal{X}, \mathcal{B}(\mathcal{X}), P_X)\) where

\[
\forall A \in \mathcal{B}(\mathcal{X}): \quad P_X(A) = \mathbb{P}(X^{-1}(A))
\]
Volume (Lebesgue Measure)

\( \lambda^N(A) \) is the volume of \( A \subset \mathbb{R}^N \)

Intuition:
- \( \lambda^1 \) is length
  - \( \lambda^1([a, b]) = b - a \)
- \( \lambda^2 \) is area
  - \( \lambda^2([a, b] \times [c, d]) = (b - a)(d - c) \)
  - \( \lambda^2([a, b] \times \{y\}) = 0 \)
- \( \lambda^3 \) is 3D-volume
- . . .

If \( A \subset \mathbb{R}^N \) is countable, then \( \lambda^N(A) = 0 \).
Three Types of Distributions

Support \( \mathcal{X} \subseteq \mathbb{R}^N, \ x \in \mathcal{X} \)

<table>
<thead>
<tr>
<th>Type</th>
<th>( \lambda^N(\mathcal{X}) )</th>
<th>( P_X({x}) &gt; 0 )</th>
<th>( H(X) )</th>
<th>( h(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disc. ( P_X^d )</td>
<td>= 0</td>
<td>&gt; 0</td>
<td>OK</td>
<td>( -\infty )</td>
</tr>
<tr>
<td>Cont. ( P_X^{ac} )</td>
<td>&gt; 0</td>
<td>= 0</td>
<td>( \infty )</td>
<td>OK</td>
</tr>
<tr>
<td>(D-C-Mixt.)</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>( \infty )</td>
<td>?</td>
</tr>
<tr>
<td>Sing. ( P_X^{sc} )</td>
<td>= 0</td>
<td>= 0</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
Joint Distribution Between Input and Output

$X$ is zero-mean, unit variance Gaussian, $Y = X^2$. 
Cantor-Distribution

“Uniform distribution on the Cantor set $\mathcal{C}$”; $\lambda(\mathcal{C}) = 0$. The CDF is continuous, but its derivative is zero a.e.
"Double-Scroll Attractor" of Chua's Circuit

\[\text{(CC-BY-SA-3.0 www.chuacircuits.com)}\]
Lebesgue-Decomposition

Let $X$ be a RV with probability distribution $P_X$, where

$$P_X(A) = \mathbb{P}(X \in A)$$

Every $P_X$ can be decomposed into the three types from the table:

$$\forall A \subseteq \mathcal{X}: \quad P_X(A) = P_X^d(A) + P_X^{ac}(A) + P_X^{sc}(A)$$