On the Correction of Linear Time-varying Systems by Means of Time-varying FIR Filters

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***Abstract*— In this paper, methods for the design of time-varying finite impulse response (FIR) filters which facilitate the correction of time-varying systems are presented. The two possible cases where an undesired time-varying system requires either preprocessing or postprocessing are considered. In particular, we show how the methods from [1], [2] can be used for the precorrection of time-varying systems. Though, these presented methods allow for the correction of systems exhibiting periodically recurring time-varying behavior, these techniques can also be applied to the non-periodic case. A mixed-signal application is presented to verify the viability of the presented precorrection case. To this end, a system model for non-uniform zero-order-hold (ZOH) signals in a digital-to-analog converter (DAC) is introduced. Finally, numerical simulations are performed to illustrate the increase of performance when the proposed precorrection is employed.**

I. Introduction

Linear time-varying systems are encountered in many technical areas, for example as a means of modeling time-varying channels [3] or signal processing blocks which exhibit time-varying behavior, i.e., time-interleaved analog-to-digital converters (ADCs) [4]. If this time-varying behavior is not desired and results in an unacceptable degradation of the overall system performance, an additional time-varying system may be introduced in the signal processing chain, preprocessing or postprocessing the undesired time-varying system. In the following, the first case will be referred to as the *precorrection* and the second case as the *postcorrection* of the time-varying behavior of an undesired system. Postcorrection structures mitigating the impact of an undesired system were presented in [5]–[8], where the error introduced by the undesired system is gradually compensated. In [9], a design technique allowing for the reconstruction of nonuniformly sampled bandlimited signals by means of a time-varying filter was presented. A related method was utilized to postcorrect general time-varying systems and a filterbank implementation of M-periodically time-varying filters was presented in [2]. Furthermore, correction schemes employing filterbank theory have been proposed in [10] and [11] alleviating the impact of nonuniformly sampling. A structure for precorrecting 2-periodically nonuniform zero-order of digital-to-analog converters has been proposed in [12], and the digital compensation of in-band spurious tones was presented in [13] for the M-periodic case.

In this paper, we present two complementary methods for the design of time-varying precorrection and postcorrection FIR filters, respectively. The proposed precorrection method can be applied to enhance the performance of non-periodically systems. The presented postcorrection method complements the precorrection case and provides the same cost function as given in [1], however, its derivation is different and presented for the sake of completeness. Furthermore, a system model for the precorrection of DAC employing nonuniform zero-order-hold signals is introduced which complements the postcorrection of ADCs in [9].

II. Precorrection of Time-varying Systems

In this section, a method for the design of the correction filter $h_n[k]$ preprocessing an undesired time-varying system $g_n[k]$ is presented. To this end, the output of the correction filter $y[n]$ is described as the convolution of the input signal $x[n]$ with its time-varying impulse response, i.e.

$$y[n] = \sum_{k=-\infty}^{\infty} h_n[k]x[n-k]. \tag{1}$$

Consecutively, the precorrected signal $y[n]$ is processed by the undesired time-varying system $g_n[k]$ resulting in

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} g_n[k]y[n-k]. \tag{2}$$

Rewriting (2) with (1), the reconstructed output of the overall system $\hat{x}[n]$ may be represented as

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} f_n[k]x[n-k] \tag{3}$$

where $f_n[k]$ is the impulse response of the two cascaded time-varying systems given as [14]

$$f_n[k] = \sum_{l=-\infty}^{\infty} g_n[l]h_{n-l}[k-l]. \tag{4}$$

The block diagram illustrating the signal flow for the cascade of the two time-varying systems is depicted in Fig. 1. Calculating the discrete-time Fourier transform (DTFT) of (4) results in a representation of the cascaded system that can be used for the design of a time-varying postcorrection filter as

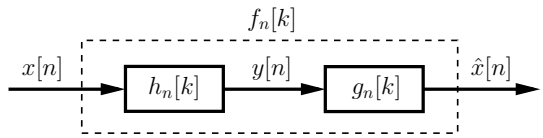

Fig. 1: Block diagram illustrating the signal flow for a cascade of two time-varying systems.

has been shown in [1]. To obtain an equivalent relationship for the precorrection case, the impulse response of the cascaded system $f_n[k]$ has to be rewritten by using the transform pair

$$g_n[k] = \hat{g}_{n-k}[k] \tag{5}$$

$$\hat{g}_n[k] = g_{n+k}[k] \tag{6}$$

as

$$f_n[k] = \sum_{l=-\infty}^{\infty} \hat{g}_{n-l}[l] h_{n-l}[k-l]. \tag{7}$$

By substituting the indexes $p = k - l$ and $l = k - p$, we get

$$f_n[k] = \sum_{p=-\infty}^{\infty} h_{n-k+p}[p] \hat{g}_{n-k+p}[k-p]. \tag{8}$$

Shifting the time index of $f_n[k]$ to $f_{n+k}[k]$ yields

$$f_{n+k}[k] = \sum_{p=-\infty}^{\infty} h_{n+p}[p] \hat{g}_{n+p}[k-p] \tag{9}$$

and applying (6) to (9) results in

$$\hat{f}_n[k] = \sum_{p=-\infty}^{\infty} \hat{h}_n[p] \hat{g}_{n+p}[k-p]. \tag{10}$$

By calculating the DTFT of (10), we obtain the frequency response of the cascaded system for each time instant n as

$$\hat{F}_n(e^{j\omega}) = \sum_{p=-\infty}^{\infty} \hat{h}_n[p] \hat{G}_{n+p}(e^{j\omega}) e^{-j\omega p}. \tag{11}$$

The relationship in (11) exhibits favorable properties. Firstly, to characterize $\hat{F}_n(e^{j\omega})$ at a given time instant, the impulse response $\hat{h}_n[p]$ can be treated as being time-invariant which is going to simplify the design of the correction filter. Secondly, for the design of an FIR filter of order K, K future states of $\hat{G}_n(e^{j\omega})$ are sufficient to determine $\hat{F}_n(e^{j\omega})$. Thirdly, though (11) can be applied when the undesired time-varying system is M-periodically time-varying, i.e. $g_n[k] = g_{n+M}[k]$, its description is not limited to this type of time-varying system since no periodically time-varying behavior of $\hat{G}_n(e^{j\omega})$ is assumed. Thus, this description can also be utilized in the non-periodic case.

In order to yield a cost function, an error function is defined as the deviation of $\hat{F}_n(e^{j\omega})$ from a desired frequency response $\hat{D}_n(e^{j\omega})$ as

$$\hat{E}_n(e^{j\omega}) = \hat{F}_n(e^{j\omega}) - \hat{D}_n(e^{j\omega}) \tag{12}$$

which is in turn used to formulate a filter design problem similar as in the time-invariant case [1], [15] as

$$\min ||\hat{E}_n(e^{j\omega})||_{\text{norm}} \quad \text{for } \omega \in \omega_D \tag{13}$$

for a desired design bandwidth ω_D and error norm. By solving the optimization problem in (13), an FIR filter design of $\hat{h}_n[k]$ for the time instant n is performed. Applying the relation in (5) to the resulting design, we obtain $h_n[k]$.

It is interesting to note that if $\hat{g}_n[k]$ exhibits an M-periodically time-varying behavior, it suffices to design M sets of filter coefficients as $h_n[k]$ will exhibit the same periodicity as the undesired system. In this case, an offline filter design of the time-varying filter can be performed.

III. Postcorrection of Time-varying Systems

In order to yield a cost function for the design of the postcorrection filter $g_n[l]$, we calculate the DTFT of (4) as [1]

$$F_n(e^{j\omega}) = \sum_{l=-\infty}^{\infty} g_n[l] H_{n-l}(e^{j\omega}) e^{-j\omega l}. \tag{14}$$

which complements the filter design for the precorrection scenario in (11). As in the precorrection case, (14) does not rely on a periodically time-varying behavior of the undesired time-varying system. Furthermore, $F_n(e^{j\omega})$ can be characterized for each time instant n by only resorting to present or past states of $H_n(e^{j\omega})$. Analogously to the precorrection case, the cost function is given as

$$E_n(e^{j\omega}) = F_n(e^{j\omega}) - D_n(e^{j\omega}) \tag{15}$$

and the filter design is obtained by minimizing (13) for $\hat{E}_n(e^{j\omega}) = E_n(e^{j\omega})$ according to a given error norm and design bandwidth.

IV. Application Example

In the following, the design of a time-varying FIR filter is presented which precorrects nonuniform sample-and-hold signals (SH) in a DAC. As an example of an SH circuit, we chose the ZOH type due to its practical importance.

A. Continuous-Time System Model

It is well-known that ZOH signals shape the analog output signal according to the continuous-time Fourier transform (CTFT) of its impulse response [16]. Moreover, spurious tones are introduced if the uniform sampling instants deviate by a time-varying jitter term $\Delta_n T$ [17]. Both effects can be modelled by the time-varying impulse response [13]

$$\tilde{a}_n(t) = T\left(u(t - \Delta_n T) - u(t - T - \Delta_{n+1} T)\right) \tag{16}$$

with T indicating the sampling period. The system employing $\tilde{a}_n(t)$ to represent the non-uniform ZOH behavior is depicted in Fig. 2. The output signal of the presented model is obtained as

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} v[k] \tilde{a}_k(t - kT) * h_{\text{id}}(t) \tag{17}$$

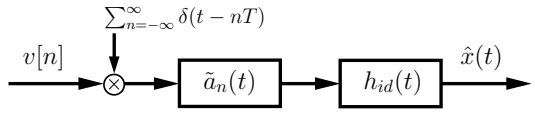

Fig. 2: DAC model employing the nonuniform ZOH model in continuous-time.

where $v[n]$ indicates the discrete-time input of the overall model. Furthermore, the impulse response of the ideal low-pass filter $h_{\text{id}}(t)$, with a cut-off frequency $0 < \Omega_D \leq \frac{\pi}{T}$, is given as [16]

$$h_{id}(t) = \frac{\Omega_D}{\pi}\text{sinc}\left(\frac{\Omega_D}{\pi}t\right). \tag{18}$$

B. Discrete-Time System Model

In order to devise an equivalent formulation of (17), we define a representation of a discrete-time filter as [16]

$$\hat{A}_n(e^{j\omega}) = \tilde{A}_n\left(j\frac{\omega}{T}\right)H_{\text{id}}\left(j\frac{\omega}{T}\right) \quad \text{for} -\pi \leq \omega < \pi \tag{19}$$

which is constituted by the discrete-time representation of the CTFT of (16), as it was presented in [13], and the CTFT of (18), with a cut-off $\Omega_D = \frac{\omega_D}{\pi}$. The output of the discrete-time filter results in

$$\hat{x}[n] = \sum_{l=-\infty}^{\infty} v[l]\hat{a}_l[n-l] \tag{20}$$

where $\hat{a}_n[k]$ is the inverse DTFT of $\hat{A}_n(e^{j\omega})$. The continuous-time signal $\hat{x}(t)$ is obtained by

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} \hat{x}[n]h_{\text{id}}(t-nT) \tag{21}$$

which represents the same continuous-time signal as obtained by the system in (17). Rewriting (20) using $k = n - l$ and $l = n - k$, we obtain

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} \hat{a}_{n-k}[k]v[n-k], \tag{22}$$

and comparing it to the time-varying system in (2) reveals the identity

$$g_n[k] = \hat{a}_{n-k}[k] \tag{23}$$

which agrees with (5). Applying this identity to the time-varying system in (2), we can represent the ZOH behavior in discrete-time as illustrated in the system model shown in Fig. 3.

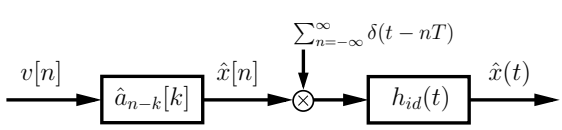

Fig. 3: DAC model employing a discrete-time ZOH model.

C. Precorrection Scheme and Filter Design

In this subsection, the overall system is presented, where the time-varying filter $h_n[k]$ precorrects the nonuniform DAC model. The design of the precorrection filter with a filter delay D is obtained by defining a design objective in terms of the desired frequency response as

$$\hat{D}_n(e^{j\omega}) = e^{-j\omega D_s} \tag{24}$$

where D_s is the accumulated delay induced by the correction and undesired time-varying system. By specifying $\hat{D}_n(e^{j\omega})$ as a delay D_s, the impact of the undesired in-band attenuation and of the non-uniform sampling are mitigated at the same time. As a consequence, the desired reconstructed output signal results in $\hat{x}[n] = x[n - D_s]$.

For the precorrection of the DAC model, we obtain for $\hat{G}_n(e^{j\omega}) = \hat{A}_n(e^{j\omega})$ in (11) the cost function

$$\hat{E}_n(e^{j\omega}) = \sum_{p=-\infty}^{\infty} \hat{h}_n[p]\hat{A}_{n+p}(e^{j\omega})e^{-j\omega p} - e^{-j\omega D_s} \tag{25}$$

by using (12) and (24). The filter design is performed by minimizing this error function for each time instant. Furthermore, the designed filter coefficients are used to continuously update the precorrection filter within the system shown in Fig. 4. To verify the proposed scheme, numerical simulations were

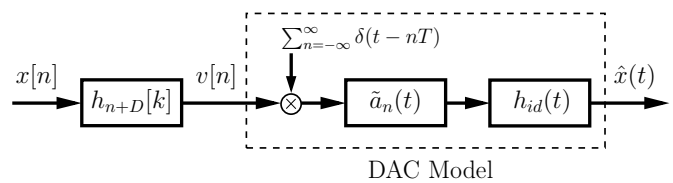

Fig. 4: System precorrecting the input signal of the DAC affected by non-uniform sampling.

performed in Matlab and the achieved correction performance was characterized in terms of the signal-to-noise ratio (SNR) given as

$$\text{SNR} = 10\log_{10}\left(\frac{\sum_{n=0}^{N-1}|x[n-D_s]|^2}{\sum_{n=0}^{N-1}|x[n-D_s]-\hat{x}[n]|^2}\right)\text{dB} \tag{26}$$

where N indicates the number of investigated samples. The non-uniform ZOH signals are generated according to a zero-mean Gaussian distribution with a standard deviation of $\sigma_z = 0.039$ and 2^{12} samples were drawn from this distribution to characterize the nonuniform ZOH behavior. The same number of samples were evaluated and a coherently sampled multitone input signal was used for the simulations. The impact of the non-uniform ZOH behavior is illustrated in Fig. 5, where the spectrum of non-bandlimited output signal is shown, when the DAC input signal is not precorrected. The depicted spectrum exhibits an in-band attenuation of up to -2 dBc for the highest frequent signal component and a multitude of spurious tones are created in-band and out-of-band. In order to suppress the spurious tones within the bandwidth $|\omega| < 0.75\pi$, a

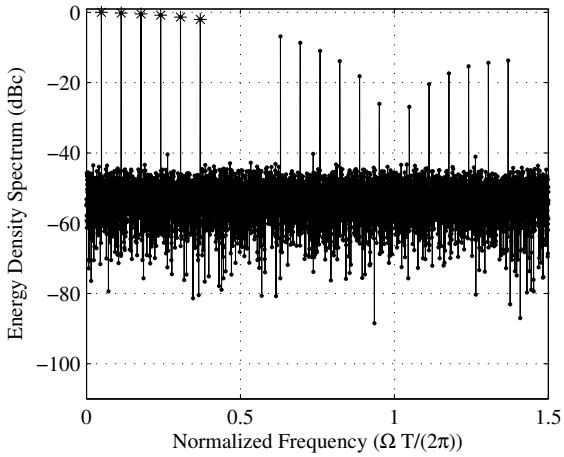

Fig. 5: Uncorrected output spectrum.

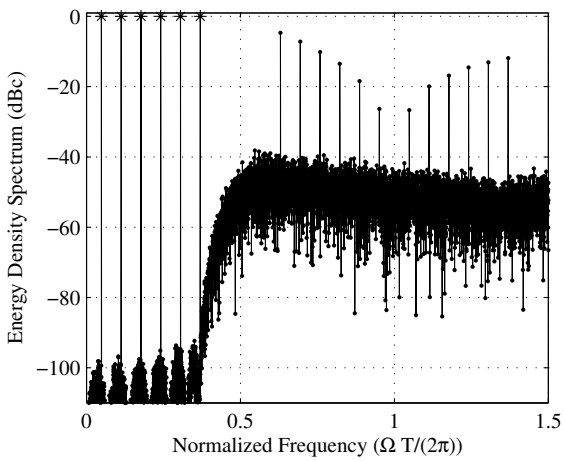

Fig. 6: Precorrected output spectrum (K=12).

time-varying precorrection filter of order $K = 12$ with a design bandwidth of $\omega_D = 0.75\pi$ was designed according to the L_2 norm. This corresponds with the bandwidth of the reconstructed signal $\hat{x}[n]$ used for the SNR calculation in (26). The precorrected output spectrum of the DAC is shown in Fig. 6 illustrating a considerable attenuation of the in-band spurious tones. Moreover, the in-band attenuation of the signal tones was compensated. The depicted signal offers a SNR value of 77.96 dB which enhances the initial SNR value for the uncorrected case by 59.92 dB.

V. Conclusion

In this paper, two methods for the design of time-varying FIR filters facilitating the precorrection and postcorrection of linear time-varying systems have been presented. These methods are able to correct non-periodically time-varying systems, however, in this case, a dedicated filter design for each time instant is required which is computationally intensive. In order to allow for an online design of the correction filter, the computational burden has to be decreased. To this end, the complexity of the optimization problem could be reduced by considering specific types of time-varying systems, e.g., systems with sparse or diagonally dominant transition matrices [18], where the filter design may be performed more efficiently. The presented methods can be used to perform an offline design for M-periodically time-varying systems.

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