Problem 1.1 (12 Points)

Given three signals $s_1(t)$, $s_2(t)$ and $s_5(t)$:

(a) Find $A$ and $B$ so that the norms $||s_1(t)|| = 1$ and $||s_2(t)|| = 1$.

(b) Find the inner product of these two signals in a linear space. What is the angle between the two signals?

(c) Find the norm of the signal $s_3(t) = s_1(t) - s_2(t)$. Sketch the signal $s_3(t)$.

(d) Find a signal $s_4(t)$ that is in the subspace spanned by $s_1(t)$ and $s_2(t)$ ($\mathcal{S} = \text{span}\{s_1(t), s_3(t)\}$) and is orthogonal to $s_3(t)$.

(e) Find and sketch the signal that lies in the subspace spanned by $s_1(t)$ and $s_2(t)$ and that is closest to $s_5(t)$.

(f) Compute and sketch the signal which represents the projection error $e_5(t)$, resulting from projecting $s_5(t)$ onto the subspace spanned by $s_1(t)$ and $s_2(t)$.

(g) Find an orthonormal basis $\{\psi_1(t), \psi_2(t)\}$ for $\mathcal{S}$. Hint: consider the geometric representation of the signals $s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t) \in \mathcal{S}$ to complete this task without cumbersome computations.

(h) Compute the coefficient vectors to represent the signals $s_1(t)$ to $s_4(t)$ in this basis (i.e. find the orthonormal basis expansions of the signals).

(i) Find the dimensionality $N$ of the signal space $\mathcal{S}_5 = \text{span}\{s_1(t), \ldots, s_5(t)\}$ as well as an orthonormal basis without cumbersome computations.
Problem 1.2 (12 Points)

Consider a binary digital communication scheme which applies pulse-position modulation (PPM). The following waveforms are used:

\[ u_1(t) = \sqrt{\frac{2E_b}{T}} \sin\left(\frac{2\pi}{T}t\right) \quad 0 \leq t \leq T \]

\[ u_2(t) = \sqrt{\frac{2E_b}{T}} \sin\left(\frac{2\pi}{T}(t - T_0)\right) \quad T_0 \leq t \leq T \]

where \( T \) is the symbol length and \( T_0 \) defines the start position of the second waveform in time domain bounded by \( 0 \leq T_0 \leq T \).

(a) Sketch the two waveforms in time domain and derive the orthonormal basis functions, as a function of the time shift \( T_0 \).

(b) Sketch the distance between the two symbols in the vector space, given by the orthonormal basis. Which condition has to hold to achieve maximum distance between the two waveforms.

(c) Find and maximize the distance between the waveforms, \( d = ||u_2(t) - u_1(t)|| \), as a function of the time shift \( T_0 \) (Hint: Use the basis derived in sub-task (a) to find the distance. To find the extreme values of the projection function the use of numerical software-tools (Octave, ...) is helpful.). Illustrate in time domain and in the vector space the two waveforms for the following cases:

1. For \( T_0 \) which achieves the maximum distance.

2. For \( T_0 \) which achieves orthogonality between the signals (Attention: There exist two cases.).

(d) Compute and illustrate the orthonormal basis for these waveforms for \( T_0 \) which achieves the maximum distances.

Problem 1.3 (9 Points)

Given the linear operator \( A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \\ 1 & 0 \end{bmatrix} \) from domain \( \mathcal{H}_0 = \mathbb{R}^2 \) to co-domain \( \mathcal{H}_1 = \mathbb{R}^3 \) such that \( y = Ax \), where \( y \in \mathbb{R}^3 \) and \( x \in \mathbb{R}^2 \). \( \alpha = 30^\circ \)

(a) What is the range \( \mathcal{R}(A) \) and the null space \( \mathcal{N}(A) \)?

(b) Is the operator \( A \) invertible, left invertible or right invertible? If yes, find the inverse / left inverse / right inverse.

(c) Find the projection operator \( P \) that approximates any vector \( y \in \mathcal{H}_1 \) within the range of \( A \). Use \( y = [1 1 1]^T \) as an example and compute \( \hat{y} \).

(d) Using numerical software (Octave, ...) show the range of \( A \) for \( x = [x_1 \ x_2]^T \), where \( x_1 \) is in the range from \(-1\) to \(1\) and \( x_2 \) is in the range from \(-2\) to \(2\) (use a spacing of 0.1. Hint: In Octave, use \texttt{plot3} without lines to visualize \( x \) and \( y \))