Lesson 4: Non-fading Memory Nonlinearities

Nonlinear Signal Processing – SS 2017

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Session contents

▶ Today:
  ▶ Nonlinear dynamical systems
  ▶ Fixed points and local stability
  ▶ Computation of trajectories
  ▶ Discrete-time nonlinear maps

▶ Next time:
  ▶ Nonlinear maps with chaotic trajectories:
    Bifurcation diagrams and Lyapunov exponents
Nonlinear dynamics – System representation

- A set of equations (nonlinear)
- In continuous time: Differential equations, \textit{flow}
  \[ \dot{x}(t) = f(x(t)) \]
- In discrete time: Difference equations, \textit{map}
  \[ x[n + 1] = f(x[n]) \]
Nonlinear dynamics – State space and trajectories

- Variables in the equations span a *phase space*
- Can be $x_1$ and $x_2$ for a 2D-eq. or also $x$ and $\dot{x}$ for a 1D eq.
- Flow: Vector field of diff. eq.: e.g. 2D system with $x$ and $y$
- Derivatives w.r.t. $x$ and $y$ in region of state space
Nonlinear dynamics – Fixed points and local stability

- **Fixed point**: \( p^* \)
- At \( x = p^* \), we have \( \dot{x} = 0 \)
- Does not mean we are automatically drawn to these points!
- Behaviour of flow/map around \( p^* \) defines *local* stability
- Find fixed points as solutions of \( \dot{x} = 0 \)
- Calculate eigenvectors/-values of Jacobian matrix (local system linearization) at these points

\[
J(x_0^*, x_1^*, \ldots, x_{N-1}^*) = \left( \begin{array}{ccc}
\frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \cdots \\
\frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \cdots \\
\vdots & \vdots & \ddots
\end{array} \right) \bigg|_{p^*=(x_0^*, x_1^*, \ldots, x_{N-1}^*)}
\]
Local Stability I

1. eigenvalues (conjugate) complex and
   - Real part positive: **instable spiral**
   - Real part negative: **stable spiral**

2. eigenvalues real and
   - Positive: **repellor**
   - Negative: **attractor**
   - Mixed: **saddle**
Local Stability II

3. Real part zero:
   ▶ analysis of local stability using linearization does not work (linearization behaves differently than NL system)
   ▶ linear behavior: ●

4. identical eigenvalues (degenerate node)
   ▶ Jacobian matrix can not be diagonalized
   ▶ Linearization does not capture behavior of the NL
   ▶ stability of linearization similar to NL

▶ Problem 4.1a as tutorial
Nonlin. dyn. – Attractors

- A set of points or a subspace in phase space, towards which trajectories converge after transients die out
- Fixed points are attractors
- Limit cycles are attractors (periodic motion)
- Quasi-periodic motion has an attractor, though same point is never visited twice
- Strange attractors: Is everything the other attractors are not
  - Set of points on which chaotic trajectories move
  - Infinite fine structure, fractal set
  - Extremely sensitive to initial conditions
  - Chaotic trajectories look random, but are not
Nonlin. dyn. – Discrete time – Maps

- In discrete time: Difference equations, \textit{map}

\[ x[n + 1] = f(x[n]) \quad \text{e.g.:} \quad x[n + 1] = 4rx[n](1 - x[n]) \]

- Fixed points \( x^* \) are defined as

\[ x[n + 1] = x[n] \]

- Stability and behavior will depend on control parameter \( r \)
Nonlinear dynamics – Bifurcation diagram

- Steady-state amplitudes as a function of control parameter
- Transients must have died out!
- E.g.: Logistic map, for $r \approx 0.75$, stable FP splits into two-point-oscillation $\rightarrow$ limit-cycle
- For $r \gtrsim 0.88$, non-periodic behavior is observed
Nonlinear dynamics – Lyapunov exponent

- Measure for sensitivity of trajectories to initial condition
- Stable fixed points: Convergence irrespective of initial point
- Instability can be local, overall amplitude still bounded!
- Lyapunov exponent $\lambda$: Like pole radius for linear systems

For $x[n + 1] = F(x[n])$:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \log |F'(x[i])|$$
Nonlinear dynamics – Lyapunov exponent and bifurcation
Estimation of Bifurcation diagram and Lyapunov exponent

- Loop over range-of-interest of control parameter $r$
- Choose an appropriate step size
- For each $r$: Iterate the map
- Throw away output in transient phase!
- Bifurcation: Record all steady-state amplitudes
- Bifurcation: Plot steady-state amplitudes over range of $r$
- Lyapunov: Estimate $\lambda$ by evaluating log of derivative of map at each $x[n]$, then average
- Lyapunov: Plot estimated $\lambda$ over $r$