# Advanced Signal Processing Seminar

## *Convex Optimization in Signal Processing*

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## Nonnegative Least Squares

Least-squares is a widely used method which finds applications in many different fields. Least-squares approximates an $D$-dimensional vector $\mathbf{x}$ using a linear combination of $K$ dictionary vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_K$. Arranging all $\mathbf{w}_k, k = 1 \ldots K$, in the columns of a dictionary matrix $\mathbf{W}$, the problem can be formulated as

$$\underset{\mathbf{h}}{\text{minimize}} \quad \|\mathbf{x} - \mathbf{W}\,\mathbf{h}\|^2$$

where $\|\|$ is the Euclidean ($\ell^2$) norm. This problem is an unconstrained (convex) quadratic problem, and the solution is well-known and given in closed form: $\mathbf{h} = \mathbf{W}^\dagger\,\mathbf{x}$, where $\mathbf{W}^\dagger$ denotes the pseudo-inverse. One often prefers a nonnegative solution of $\mathbf{h}$, i.e. one is interested in the nonnegative least-squares problem (NNLS):

$$\begin{aligned} \underset{\mathbf{h}}{\text{minimize}} \quad & \|\mathbf{x} - \mathbf{W}\,\mathbf{h}\|^2 \\ \text{subject to} \quad & \mathbf{h} \geq 0 \end{aligned} \tag{1}$$

where $\geq$ has to be understood element-wise. This problem, although convex, does not have an closed-form solution any more. However, in [3] an iterative active-set algorithm was proposed to solve the NNLS problem.

## Nonnegative Matrix Factorization

Nonnegative matrix factorization (NMF) [1, 2] aims to approximate an element-wise nonnegative $D \times N$ matrix $\mathbf{X}$ by the product of two element-wise nonnegative matrices $\mathbf{W}$ and $\mathbf{H}$, i.e. we want to find $\mathbf{W}$ and $\mathbf{H}$, such that $\mathbf{X} \approx \mathbf{W}\,\mathbf{H}$, where $\mathbf{W}$ is a $D \times K$ matrix and $\mathbf{H}$ is a $K \times N$ matrix. $K$ is called the approximation rank, and typically one chooses $K$ such that $D\,K + K\,N \ll D\,N$ (i.e. the factorization is *compressive*). NMF is used since about 20 years in various fields, such as image analysis, gene expression analysis, speech and audio processing, to name but a few. The NMF problem can be formulated as

$$\begin{aligned} \underset{\mathbf{W},\mathbf{H}}{\text{minimize}} \quad & \|\mathbf{X} - \mathbf{W}\,\mathbf{H}\|_F^2 \\ \text{subject to} \quad & \mathbf{W} \geq 0 \\ & \mathbf{H} \geq 0 \end{aligned} \tag{2}$$

where $\geq$ has to be understood element-wise. Here $\|\|_F$ denotes the Frobenius norm: $\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$. Although this problem is convex in *either* $\mathbf{W}$ *or* $\mathbf{H}$, it is *not* convex in $\mathbf{W}$ *and* $\mathbf{H}$ jointly (and in fact proven to be NP-hard). Note that optimizing (2) w.r.t. $\mathbf{H}$ decomposes into $N$ independent problems of the form (1), i.e. the columns of $\mathbf{H}$ can be solved independently. Similarly, optimizing w.r.t. $\mathbf{W}$ decomposes into $D$ independent problems of the form (1), i.e. the rows of $\mathbf{W}$ can be solved independently (after transposing $\mathbf{X}$, $\mathbf{W}$ and $\mathbf{H}$). In order to find a local optimum of (2), one can solve alternately w.r.t. $\mathbf{W}$ and $\mathbf{H}$, which is called alternating nonnegative least-squares (ANLS) [4, 5].

**To do**

- Review and present the active-set algorithm described in [3].
- Review and present NMF using the ANLS scheme [4, 5].

- Implement NMF using the ANSL scheme (e.g. in Matlab). For NNLS you can use the implemented function `lsqnonneg`.
- Download the CBCL face image database:
  `http://cbcl.mit.edu/software-datasets/FaceData2.html`
- Load the training set, containing 2429 face images of of size $19 \times 19$ pixels. Rearrange the images into 361-dimensional vectors, such that you obtain a $361 \times 2429$ data matrix.
- Apply your NMF algorithm to this data matrix. Choose an appropriate approximation rank $K$ (not too large).
- Re-interpret the columns of the obtained matrix $\mathbf{W}$ (the basis vectors) as images. Show them in your presentation. Interpret the results and compare to [1].
- Hand in your presentation slides and your code, which will be made available via our homepage.

# References


[1] Daniel D. Lee and H. Sebastian Seung, Learning the Parts of Objects by Nonnegative Matrix Factorization, *Nature*, vol. 401, 788–791, 1999.

[2] Daniel D. Lee and H. Sebastian Seung, Algorithms for Non-negative Matrix Factorization, *NIPS*, 2001.

[3] Lawson, C. and Hanson, R., *Solving Least Squares Problems*, Prentice-Hall; 1974.

[4] Lin, C.J., Projected Gradient Methods for Nonnegative Matrix Factorization, *Neural Computation*, vol. 19, 2756–2779, 2007.

[5] Kim, H. and Park, H., Non-Negative Matrix Factorization Based on Alternating Non-Negativity Constrained Least Squares and Active Set Method, *SIAM Journal on Matrix Analysis and Applications*, vol. 30, 713–730, 2008.