Linear Programming Algorithms for Sparse Filter Design

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Outline

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Introduction

Efficient implementation of discrete-time filters

- Complexity reduction of filters
  - reduction of number of arithmetic operations
  - restriction of arithmetic operations to those with fast execution on dsp
  - our approach: sparse filters

Sparse filters

- filter with few nonzero impulse response coefficients
- pro: arithmetic operations for zero-valued coefficients can be omitted altogether

Sparse Filter Design

Direct design

- complex and difficult task
- time and ressource consuming

Complexity reduction of given filter

- start with filter with desired response
- zero-forcing of some coefficients
- optimize remaining nonzero coefficients to achieve minimum deviation of the resulting response and the desired response
- feasible result: sparse filter with a deviation below certain threshold

Sparse Filter Design

Problems and Challenges

- larger number of possible combinations of zero-valued and nonzero coefficients
- still computationally difficult task
- in theory guaranteed to yield a maximal sparse filter (alls sparsity problems are considered)

Solution: use approximations

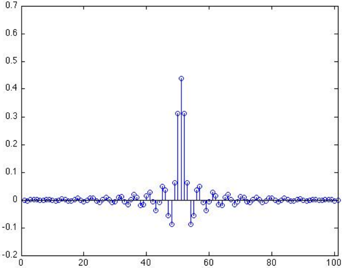
- goal: reasonably sparse filters
- resulting filter is not necessarily optimal
- idea: just consider a certain subset of possible combinations

Outline

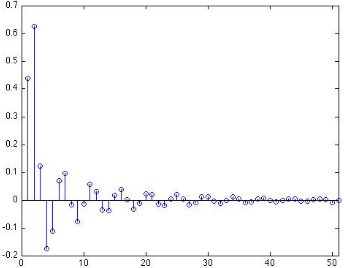
Problem Formulation

Given Filter: Type I linear-phase FIR

$$b_0 = h[m]$$
$$b_m = 2h[M-n] = 2h[M+n], \qquad n = 1, 2, ..., M$$



h



b

Problem Formulation

Frequency response

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega M}$$

$$A(e^{j\omega}) = \sum_{n=0}^{M} b_n \cos(n\omega)$$

$$W(\omega)|A(e^{j\omega}) - H_d(e^{j\omega})| \leq \delta_d \qquad \forall\omega$$

Problem Formulation

optimization problem

$$\min_{b_0,\dots,b_M} \quad \|\mathbf{b}\|_0$$
$$\text{s.t.} \quad W(\omega)|A(e^{j\omega}) - H_d(e^{j\omega})| \leq \delta_d \qquad \forall\omega$$

Successive Thinning Algorithm

$$\min_{\{b_n\},\delta} \quad \delta$$
$$\text{s.t.} \quad W(\omega)|A(e^{j\omega}) - H_d(e^{j\omega})| \leq \delta \qquad \forall\omega$$
$$b_n = 0 \qquad \forall n \in \mathcal{Z}^{(i)}$$

Successive Thinning Algorithms

discrete Frequencies

$$\begin{aligned} \min_{\{b_n\},\delta} \quad & \delta \\ \text{s.t.} \quad & W(\omega_k)\left|\sum_{n=0}^{M} b_n \cos(n\omega_k) - H_d(e^{j\omega_k})\right| \leq \delta, \quad k = 1, ..., K \\ & b_n = 0 \qquad \forall n \in \mathcal{Z}^{(i)} \end{aligned}$$

general linear program in standard form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{f}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned}$$

Successive Thinning Algorithms

$$\mathbf{A}_{kn} = W(\omega_k)\cos(n\omega_k), \quad k = 1, ..., K, n = 0, ..., M$$
$$\mathbf{h}_k = W(\omega_k)H_d(e^{j\omega_k}), \quad k = 1, ..., K$$
$$\mathbf{e}_k = 1, \quad k = 1, ..., K$$

resulting linear program

$$\begin{aligned} \min_{\delta,\mathbf{b}} \quad & \delta \\ \text{s.t.} \quad & \begin{bmatrix} -\mathbf{e} & \mathbf{A}_{kn} \end{bmatrix} \begin{bmatrix} \delta \\ \mathbf{b} \end{bmatrix} \leq \mathbf{h} \\ & \begin{bmatrix} -\mathbf{e} & -\mathbf{A}_{kn} \end{bmatrix} \begin{bmatrix} \delta \\ \mathbf{b} \end{bmatrix} \leq -\mathbf{h} \\ & b_n = 0 \qquad \forall n \in \mathcal{Z}^{(i)} \end{aligned}$$

Successive Thinning Algorithms

corresponding dual problem

$$
\begin{aligned}
\max_{\{p_k^+\},\{p_k^-\}} \quad & \sum_{k=1}^{K} W(\omega_k) H_d(e^{j\omega_k})(p_k^+ - p_k^-) \\
\text{s.t.} \quad & \sum_{k=1}^{K} W(\omega_k) \cos(n\omega_k)(p_k^+ - p_k^-) \qquad \forall n \in \mathcal{N}^{(i)} \\
& \sum_{k=1}^{K} (p_k^+ - p_k^-) = 1 \\
& p_k^+ \geq 0, p_k^- \geq 0, \qquad k = 1, ..., K
\end{aligned}
$$

Selection Rules for selecting next Zero coefficient

Two possible heuristics

- Minimum-Increase Rule

$$m^{(i)} = \arg \min_{p \in \mathcal{C}^{(i)}} \delta^{(i)}(p)$$

- Smallest-Coefficient Rule

$$m^{(i)} = \arg \min_{n \in \mathcal{N}^{(i)}} \left| b_n^{(i)} \right|$$

Efficiency Enhancements

$$\tilde{\mathcal{Z}} = \mathcal{Z} \cup \{m\}$$
$$\mathcal{N} = \tilde{\mathcal{N}} \cup \{m\}$$

exploid sparsity of **b**

$$\begin{aligned} \min_{\delta,\mathbf{b}} \quad & \delta \\ \text{s.t.} \quad & \begin{bmatrix} -\mathbf{e} & \mathbf{A}_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} \delta \\ \mathbf{b}_{\mathcal{N}} \end{bmatrix} \leq \mathbf{h} \\ & \begin{bmatrix} -\mathbf{e} & -\mathbf{A}_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} \delta \\ \mathbf{b}_{\mathcal{N}} \end{bmatrix} \leq -\mathbf{h} \end{aligned}$$

Outline

Minimum 1-Norm Design

Developement of an algorithm

- smallest coefficient method: in each loop iteration smallest nonzero coefficient set to zero, subsequent optimization
- similar strategy for filters with many small coefficients
- two stage implementation
- first stage: optimize filter for large number of small coefficients
- define a desired number of $\mathcal{J}$ coefficients allowed to be nonzero
- stage two: set $M + 1 - \mathcal{J}$ smallest coefficients to zero, re-optimize other coefficients

Stage 1

Optimization objective

- minimize 1-norm of impulse response vector
- thereby, promote designs with many small-valued coefficients
- idea: coefficients with small values in this step will be identical with those still yielding a feasible result when set to zero

Stage 1

definition of 1-norm

$$\|\mathbf{b}\|_1 = \sum_{n=0}^{M} |b_n|$$

resulting opimization problem

$$\begin{aligned} \min_{\{b_n\}} \quad & \|\mathbf{b}\|_1 \\ \text{s.t.} \quad & W(\omega)|A(e^{j\omega}) - H_d(e^{j\omega})| \geq \delta_d \quad \forall \omega \in \mathcal{S} \\ & b_n = 0 \quad \forall n \in \mathcal{Z} \end{aligned}$$

Stage 1

resulting linear program

$$\begin{aligned} \min_{\mathbf{m},\mathbf{b}} \quad & [1, 1, \cdots, 1]\mathbf{m} \\ \text{s.t.} \quad & \mathbf{A}_{\mathcal{N}}\mathbf{b}_{\mathcal{N}} \geq \mathbf{h} - \delta_d \mathbf{e} \\ & -\mathbf{A}_{\mathcal{N}}\mathbf{b}_{\mathcal{N}} \geq -\mathbf{h} - \delta_d \mathbf{e} \\ & \mathbf{m} + \mathbf{b}_{\mathcal{N}} \geq 0 \\ & \mathbf{m} - \mathbf{b}_{\mathcal{N}} \geq 0 \\ & \mathbf{m} \geq 0 \end{aligned}$$

m is strictly positive vector constrained to bound magnitude of coefficients b_n

decreasing $\sum m$ should decrease coefficients b_n

Stage 2

set a certain number of $M + 1 - \mathcal{J}$ coefficients to zero
only $\mathcal{J}$ coefficients of the impulse response have nonzero values

$$\begin{aligned} \min_{\{b_n\},\delta} \quad & \delta \\ \text{s.t.} \quad & W(\omega_k)|A(e^{j\omega_k}) - H_d(e^{j\omega_k})| \le \delta \\ & \qquad k = 1, \cdots, K \\ & b_n = 0 \quad \forall n \notin \mathcal{N}_{\mathcal{J}} \end{aligned}$$

$\mathcal{N}_{\mathcal{J}}$ denotes index set of $\mathcal{J}$ nonzero coefficients in b_n

Stage 2

Perform binary search to determine smallest $\mathcal{J}$ still yielding a feasible result

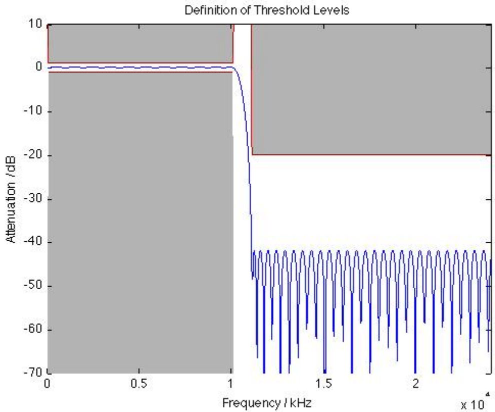
- $\mathcal{J}_0$ feasible: all $\mathcal{J} > \mathcal{J}_0$ are feasible
- $\mathcal{J}_0$ not feasible: all $\mathcal{J} < \mathcal{J}_0$ are not feasible

Outline

Definition of Thresholds

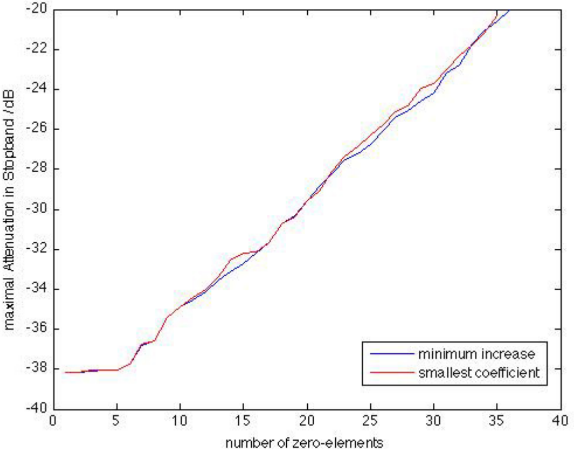
Ex: LP Filter

Passband:	$\pm$ 1 dB
Stopband attenutaion:	$<$ 20 dB



Error in Stopband over number of Zero-Elements

LP Filter, $N = 100$



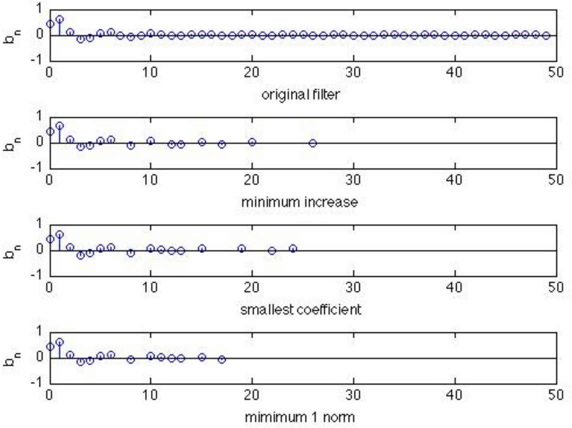
Impulse response

LP Filter, $N = 100$

stopband threshold level [dB]	algorithm	non-zero weights
-30	min. increase	32
	smalllest coeff.	32
	min. 1-norm	30
-25	min. increase	23
	smalllest coeff.	24
	min. 1-norm	24
-20	min. increase	15
	smalllest coeff.	16
	min. 1-norm	14

Nonzero elements of Impulse response

TP Filter, $N = 100$, threshold level = -20 dB



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Conclusion

Linear Program Algorithms for sparse filter design results in

- decrease of computational complexity
- tradeoff between complexity and filter accuracy
- may not yield optimal solution

Thank you for your attention!

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