Signal Processing and Speech Communication Lab.
Graz University of Technology

AS

Adaptive Systems—Homework Assignment 2

Name(s) Matr.No(s).

The analytical part of your homework (your calculation sheets) **as well as** the MATLAB simulation protocol have to be delivered as **hard copy** to our mailbox at Inffeldgasse 16c, ground floor, no later than **2018/1/19, 15:00**. Use a printed version of **this entire document** as the title pages and fill in your **name(s) and matriculation number(s)**. Submitting your homework as a LaTeX document can earn you **up to 3 points!**
Your MATLAB programs (`*.m` files) and the simulation protocol (in pdf format!) have to be submitted via **e-mail** to the address `hw2.spsc@tugraz.at` no later than **2018/1/19**. The subject of the e-mail consists of the assignment number and your matriculation number(s) **"Assignment2, MatrNo1, MatrNo2"**. You have to zip (or tar) all your homework files to one single file with the name `Assignment2_MatrNo1_MatrNo2.zip`, e.g., `Assignment2_01312345_01312346.zip`, which has to be attached to the e-mail.
Please make sure that your approaches, procedures, and results are clearly presented. Justify your answers!

Analytical Problem 2.1 (5 Points)—Gradient Algorithm

Consider the following linear filtering problem:

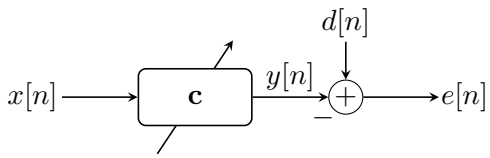

Figure 1: Wiener Filter

A variation of the normal gradient search is the *coefficient–leakage gradient search* algorithm that uses the coefficient update rule

$$\mathbf{c}[n] = (1 - \mu\alpha)\mathbf{c}[n-1] + \mu(\mathbf{p} - \mathbf{R}_{xx}\mathbf{c}[n-1])$$

where α is the leakage coefficient ($0 \leq \alpha < 1$) and μ the step size.

(a) Assume that the algorithm converged. Where does the algorithm converge to?.

(b) Transform the update equation such that it adapts the misalignment vector $\mathbf{v}[n] = \mathbf{c}[n] - \mathbf{c}_{MSE}$.

(c) Apply a unitary coordinate transform such that the transformed components of the misalignment vector are decoupled.

(d) Use the decoupled update equation of the misalignment vector and express it as a function of time n and the initial transformed misalignment vector $\tilde{\mathbf{v}}[0]$.

(e) Find an expression of the time constants for each of the decoupled coefficients. What is the influence of the leakage parameter α?

Least–Mean–Square (LMS) Adaptive Filters

In contrast to the gradient algorithm where a *deterministic* gradient is used to recursively compute the Wiener filter, the LMS algorithm uses a *stochastic* approximation of the gradient and is thus part of the class of *stochastic gradient algorithms*.
As shown in the problem class, the coefficient update for the LMS equation can be derived from the coefficient update equation of the gradient algorithm by using an instantaneous estimate of the gradient. The result is a more or less random update of the coefficients form one iteration cycle to the next one, explaining the use of the term *stochastic* gradient.
This standard update equation for the LMS algorithm is

$$\text{LMS}: \quad \mathbf{c}[n] = \mathbf{c}[n-1] + \mu e[n]\mathbf{x}[n] \tag{1}$$

Three popular variations are

$$\text{normalized LMS}_1: \quad \mathbf{c}[n] = \mathbf{c}[n-1] + \frac{\mu}{\|\mathbf{x}[n]\|^2} e[n]\mathbf{x}[n] \tag{2}$$

$$\text{normalized LMS}_2: \quad \mathbf{c}[n] = \mathbf{c}[n-1] + \frac{\mu}{\|\mathbf{x}[n]\|^2 + \alpha} e[n]\mathbf{x}[n] \qquad \alpha > 0 \tag{3}$$

$$\text{leaky LMS}: \quad \mathbf{c}[n] = (1-\mu\alpha)\mathbf{c}[n-1] + \mu e[n]\mathbf{x}[n] \qquad 0 \le \alpha < 1 \tag{4}$$

The following problems use these four update equations to examine their properties and differences.

MATLAB Problem 2.2 (8 Points)—LMS Variations

(a) Shortly describe the main differences of the four given update equations (Eq. 1-4) concerning the usage of the old coefficients and the coefficient update. What problems are treated by these different LMS variations?

(b) Write a Matlab function according to the given header that implements all four LMS versions (Eq. 1-4). The integer `OPTS` chooses which LMS version is used to compute the coefficients.

```
function [y,e,c] = v_lms(x,d,N,mu,alpha,OPTS,c0)
% INPUTS:
% x ....... input signal vector
% d ....... desired output signal (of same length as x)
% N ....... number of filter coefficients
% mu ...... step size parameter
% alpha ... algorithm dependent parameter
% OPTS .... set "1" for NLMS, "2" for coefficient-leakage LMS, "0" for LMS
% c0 ...... initial coefficient vector (optional; default all zeros)
% OUTPUTS:
% y ....... output signal vector (of same length as x)
% e ....... error signal vector (of same length as x)
% c ....... coefficient matrix (N rows, number of columns = length of x)
```

Test your implementation with the script `lms_test.m` provided on the course webpage! In addition to that, write down what you observe in the five generated plots.

MATLAB Problem 2.3 (15 Points)—LMS Performance Analysis

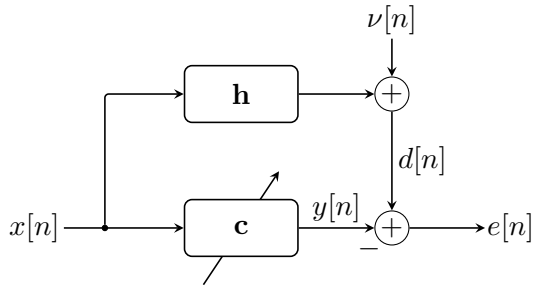

Figure 2: System identification in a noisy environment.

For the system identification problem shown in Figure 2 we want to determine the convergence time constants τ_i using the *ensemble-averaged*[1] misalignment vector

$$\mathbf{v}[n] = \mathbf{c}[n] - \mathbf{c}_{MSE} = \mathbf{c}[n] - \mathbf{h}.$$

The input signal $x[n]$ is a zero-mean, white Gaussian process with unit variance. The desired signal $d[n]$ is corrupted by additive white Gaussian noise with zero mean and variance $\sigma_\nu^2 = 0.008$. The impulse response of the unknown system is given as

$$\mathbf{h} = [0.6, 0.2, 0.4]^T$$

and $N = \dim(\mathbf{c}[n]) = \dim(\mathbf{h})$.
For the following tasks, use your function from Problem 2.2. Call it multiple times and use the coefficient matrices to compute the ensemble averages of the misalignment vector. Initialize your coefficiens with a zero vector. Write a Matlab/Octave script to plot

$$\ln \frac{\mathbb{E}\{v_k[n]\}}{\mathbb{E}\{v_k[0]\}}$$

as a function of time n for all components k of the vector. Your script should automatically determine the time constants τ_k and print them in the legend of the plot. On top of the curves, plot the logarithm of the MSE as a thick line, i.e.,

$$\ln \frac{\mathbb{E}\{e^2[n]\}}{\mathbb{E}\{e^2[0]\}}.$$

(a) Investigate the effect of the step-size parameter μ for both the LMS (Eq. 1) and the NLMS$_1$ algorithm (Eq. 2). Use, e.g., $\mu = \{0.0001, 0.001, 0.01, 1\}$. Are the algorithms stable in all these cases?

(b) Now set the step size to $\mu = 0.001$ and vary the variance of the input signal $x[n]$. Use $\sigma_x^2 = \{0.2, 1, 5\}$. What is the difference between the LMS and the NLMS?

(c) Let $x[n]$ be a zero-mean, unit variance, white input process $w[n]$ filtered by a two-tap moving average filter, i.e.,

$$x[n] = \frac{w[n]}{2} + \frac{w[n-1]}{2}.$$

[1] For every considered time instant n we average over data obtained from independent trials, i.e., using multiple realizations of the input and noise processes.

Use again a small step size of $\mu = 0.001$ and plot convergence behavior of the misalignment vector. What can you observe? Can you still determine the time constants? What can you say about the covergence behavior of the MSE as a function of time?

(d) In all previous tasks, how did the MSE behave *after* convergence?

(e) BONUS (4 Points): For all evaluated scenarios compute the *excess MSE*. What can you observe?

Analytical Problem 2.4 (5 Points)—Average Behaviour of the LMS

The convergence properties of the LMS algorithm can only be examined *on average*. Using the assumption of convergence on average, i.e.

$$\mathbb{E}\{\mathbf{c}[n]\} = \mathbb{E}\{\mathbf{c}[n-1]\} = \mathbb{E}\{\mathbf{c}_\infty\}\,,$$

examine the convergence behaviour of the LMS (Eq. 1) and the coefficient–leakage LMS (Eq. 4) in a noisy system identification setup (Figure 2). Compare the results and describe the properties of the different update equations.
Hint: Compute $\mathbb{E}\{\mathbf{c}[n]\}$ and express the converged coefficients assuming a white input signal $x[n]$ with variance σ_x and uncorrelated noise $\nu[n]$.

Analytical Problem 2.5 (3 Points)—Bonus: Coefficient-Leakage LMS

The cost function for the leaky LMS is given as

$$J[n] = e^2[n] + \alpha \left\|\mathbf{c}[n-1]\right\|^2$$

Use it to derive the coefficient update equation for the coefficient–leakage LMS algorithm as given in Eq. 4.