Signal Processing and Speech Communication Lab.
Graz University of Technology

AS

Adaptive Systems—Homework Assignment 1

Name(s) Matr.No(s).

The analytical part of your homework (your calculation sheets) **as well as** the MATLAB simulation protocol have to be delivered as **hard copy** to our mailbox at Inffeldgasse 16c, ground floor, no later than **2017/12/11**. Use a printed version of **this entire document** as the title pages and fill in your **name(s) and matriculation number(s)**. Submitting your homework as a LaTeX document can earn you **up to 3 points!**
Your MATLAB programs (`*.m` files) and the simulation protocol (in pdf format!) have to be submitted via **e-mail** to the address `hw2.spsc@tugraz.at` no later than **2017/12/11**. The subject of the e-mail consists of the assignment number and your matriculation number(s) **"Assignment1, MatrNo1, MatrNo2"**. You have to zip (or tar) all your homework files to one single file with the name `Assignment1_MatrNo1_MatrNo2.zip`, e.g., `Assignment1_01312345_01312346.zip`, which has to be attached to the e-mail.
Please make sure that your approaches, procedures, and results are clearly presented. Justify your answers!

Analytical Problem 1.1 (10 Points)—Theory

Linear Algebra
With the step from the continuous to the discrete domain, signal processing in general profits from many readily available results taken from linear algebra and matrix theory.

(a) Answer the following statements with true of false, including an appropriate proof (sometimes, a counter example is sufficient).

(i) Every Toeplitz matrix is positive-definite.

(ii) Every Toeplitz matrix is a square matrix.

(iii) A matrix $\mathbf{A}$ is called positive-definite if at least one eigenvalue λ_i of $\mathbf{A}$ is zero.

(b) Perform the following tasks concerning matrix manipulations.

(iv) Assume that you have a matrix $\mathbf{A}$ whose columns vectors $\mathbf{a}_i$ are an orthonormal basis. How can you project a vector $\mathbf{x}$ onto that subspace (i.e. the column space of $\mathbf{A}$)? Assume that $\mathbf{A}$ is a (3×3) matrix.

(v) Consider an arbitrary vector $\mathbf{x}$. Write down matrices for

- reversing
- stretching
- rotation by an arbitrary angle φ

Include a simple example and sketch the resulting vectors.

(vi) Show that $\det(a\mathbf{A}) = a^N \det(\mathbf{A})$ where $\mathbf{A}$ is an $N \times N$ matrix and a is a scalar constant.

Principles of Optimization
Very often we will be confronted by the task of finding an optimal solution to a certain problem by minimizing some objective function. Different objective functions might lead to different optimal solutions, while the steps taken to find these solutions are often very similar.

(c) Consider an optimization problem of an objective function $f(\mathbf{x})$, i.e. we want to find the vector $\mathbf{x}$ which minimizes the objective function

$$\mathbf{x}_{\text{opt}} = \underset{\mathbf{x}}{\operatorname{argmin}}\, f(\mathbf{x})$$

(i) What are the two optimality conditions which must be satisfied $\mathbf{x}_{\text{opt}}$ by the solution?

(ii) Under the assumption, that $\mathbf{x}_{\text{opt}}$ is a minimizer, apply a second–order Taylor–series expansion of $f(\mathbf{x})$ in a sufficiently small neighborhood around $\mathbf{x}_{\text{opt}}$ and explain the two optimality conditions.

Wiener Filters

In the class of filters, the Wiener filter yields the optimum solution in the *mean–square error* sense, while requiring stationarity of the involved signals, as well as knowledge of their correlation functions. A popular approach for derivation is based on the gradient of the *mean–square error* cost function $J_{\mathrm{MSE}}(\mathbf{c})$.

(d) An alternative approach to derive the Wiener filter is the so–called *orthogonality principle*.

(i) Derive the orthogonality principle

(ii) Derive the Wiener–Hopf solution from (i)

(e) A straightforward way to compute the optimal filter coefficients is to invert the autocorrelation matrix $\mathbf{R}_{xx}$.

(i) What are the two major problems with this approach?

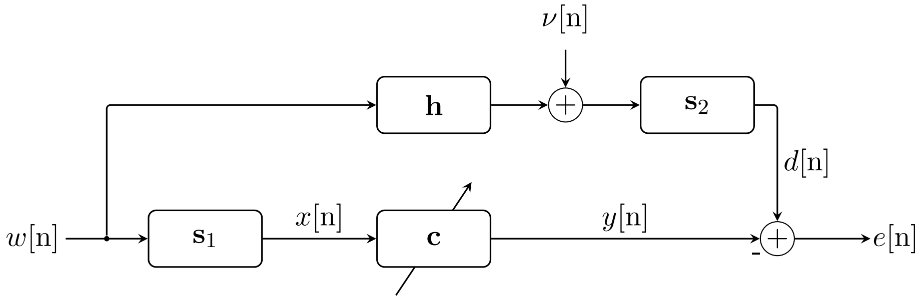
(ii) Assume that the input signal is $x[n] = e^{j\theta n}$, i.e. it has a line spectrum with a single line ($L = 1$). Compute the $N \times N$ autocorrelation matrix $\mathbf{R}_{xx}$ using $N > L$ for the noisy signal

$$x[n] = e^{j\theta n} + w[n],$$

where $w[n]$ is zero-mean Gaussian noise with non-zero variance σ_w^2. Check if the matrix is inveritble. Try to reason why $\mathbf{R}_{xx}$ in fact is invertible. What happens for $\sigma_w^2 \to 0$?

Analytical Problem 1.2 (12 Points)—System Identification with Realistic Sensors

Consider the following system identification problem:



Assume that $w[n]$ and $\nu[n]$ are independent, jointly stationary, white noise processes with variances $\sigma_w^2 = 1$ and $\sigma_\nu^2 = 0.3$, respectively. We want to identify the unknown LTI system $H(z) = 3 + z^{-1}$ using a second-order adaptive filter $C(z)$ (i.e., $N = 3$ and $\mathbf{c} = [c_0, c_1, c_2]^T$). Unfortunately, we cannot assume that our adaptive filter has direct access to $x[n]$ or $d[n]$, but only to filtered versions of these, which are picked up via sensors $S_1(z)$ and $S_2(z)$.
For all the following cases, determine

1. the values of the auto-correlation sequence $r_{xx}[k]$ and the autocorrelation matrix $\mathbf{R}_{xx}$,
2. the values of the cross-correlation vector $\mathbf{p} = \mathbb{E}\left(d[n]\mathbf{x}[n]\right)$,
3. the optimal coefficient vector $\mathbf{c}_{\text{opt}}$ in the sense of a minimum mean-squared error, i.e. $\mathbf{c}_{\text{opt}} = \operatorname{argmin}_{\mathbf{c}} J(\mathbf{c})$ where $J(\mathbf{c}) = \mathbb{E}\left(|e[n]|^2\right)$, and
4. the minimum mean-squared error $J_{\min} = J(\mathbf{c}_{\text{opt}})$.

Answer if you can identify the system correctly, and if so, why this is the case. If not, explain why system identification fails. Also, describe all your observations! **Hint:** In some cases it may be helpful to redraw the signal model by taking the linearity of the systems into account.

(a) Let $S_1(z) = S_2(z) = 1$.

(b) Let $S_1(z) = S_2(z) = 1 + 0.5z^{-1}$.

(c) Let $S_1(z) = 1$ and $S_2(z) = z^{-1}$.

(d) Let $S_1(z) = 1$ and $S_2(z) = 1 + z^{-2}$. Can you at least identify the *cascade* $H(z)S_2(z)$ correctly?

MATLAB Problem 1.3 (13 Points)—Least-Squares Tracking of a Time-Varying System

Consider the system identification problem for a time-varying impulse response. Assume that the desired signal follows

$$d[n] = \mathbf{h}^{\mathrm{T}}[n]\mathbf{x}[n] + \nu[n]$$

where $\mathbf{h}[n]$ indicates that the impulse response now depends on the current time n.

(a) Draw the block diagram of the problem, labeling all the important signals and blocks.

(b) Write a MATLAB function that computes the optimum filter coefficients in the sense of *least squares* according to the following specifications:

```
function c = ls_filter( x, d, N)
% x ... input signal
% d ... desired output signal (of same length as x)
% N ... number of filter coefficients
```

(c) For the 'unknown' system, implement a filter with the following time-varying 3-sample impulse response:

$$\mathbf{h}[n] = \begin{bmatrix} 1 \\ 0.98^n \\ 0.125 \cdot \cos\theta n \end{bmatrix},$$

where $\theta = \frac{4\pi}{1000}$. Visualize this time-varying impulse response in a single plot.

(d) Assume now, that $\nu[n] = 0$. Generate 1000 samples (for $n = 0 \dots 999$) of an input signal drawn from a stationary white noise process with zero mean and variance $\sigma_x^2 = 1$. Compute the output $d[n]$ of the system, under the condition that all delay elements are initialized with zeros (i.e., $x[n] = 0$ for $n < 0$).
The adaptive filter has 3 coefficients ($N = 3$). By calling the MATLAB function `ls_filter` with length-M segments of both $x[n]$ and $d[n]$, the coefficients of the adaptive filter $\mathbf{c}[n]$ for $n = 0 \dots 999$ can be computed. Note, we then obtain $\mathbf{c}[n] = \operatorname{argmin}_{\mathbf{c}} J(\mathbf{c}, n)$ where we may rewrite the cost function as $J(\mathbf{c}, n) = \sum_{k=n-M+1}^{n} |e[k]|^2$. For segment lengths of $M \in \{20, 50\}$, create plots comparing the elements of the coefficient vector $c_j[n]$ to the elements of the impuls response vector $h_j[n]$. Compare and discuss your results and explain the effects of M.

(e) Repeat task **(d)** for $\nu[n]$ being a zero-mean white noise process with variance $\sigma_\nu^2 = 0.02$.

(f) Repeat task **(d)** for $\nu[n]$ being generated by

$$\nu[n] = \frac{1}{3}x[n] + \frac{1}{5}x[n-2].$$

MATLAB Problem 1.4 (8 Points)—Bonus: Weighted Least Squares

This task also deals with linear filtering as introduced in the last problem. Consider that observations of $x[n]$ and $d[n]$ for $n = 0 \dots M$ are given.

(a) Derive the optimum filter coefficients $\mathbf{c}$ in the sense of a weighted least squares, i.e., find $\mathbf{c}_{\text{opt}} = \operatorname{argmin}_{\mathbf{c}} J(\mathbf{c}, n)$ where the cost function is

$$J_{\text{w}}(\mathbf{c}, n) = \sum_{k=n-M+1}^{n} w[k-n] \cdot |e[k]|^2.$$

Use vector/matrix notation!

(b) Explain the effect of the weighting and answer for what scenario(s) such a weighting may be sensible. Discuss how to select M for a stationary, a slowly time-varying, or a quickly time-varying scenario.

(c) Use your Matlab function from Problem 1.3 and expand it to implement the above cost function $J_w(\mathbf{c}, n)$ for the weighted LS algorithm.

(d) Evaluate and compare the performance using the weighting functions

- (i) $w[n] = n + M$
- (ii) $w[n] = \lambda^{-n}$ for

and using the same filter coefficients as in Problem 1.3, but $\theta = \frac{2\pi}{100}$ and $M = 50$ (also use the noisy observation case with $\sigma_\nu^2 = 0.02$). Try to find a suitable value for λ that accomplishes the tracking.